MATH2132 Test1 Solutions

Values

5 **1.** Find the limit of the following sequence of functions, if it exists.

$$f_n(x) = \frac{n^2x^2+1}{2n^2x^2+4}, \qquad -1 \le x \le 1.$$

$$\lim_{n\to\infty} \frac{n^2x^2+1}{2n^2x^2+4} = \lim_{n\to\infty} \frac{x^2+1/n^2}{2x^2+4/n^2} = \frac{x^2}{2x^2} = \frac{1}{2}, \quad \text{provided } x \neq 0$$

When $x = 0$,

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \frac{1}{4} = \frac{1}{4}.$$

Thus,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1/2, & -1 \le x \le 1,\ x \neq 0 \\ 1/4, & x = 0. \end{cases}$$

4 **2.** You are given that the series of constants $\sum_{n=1}^{\infty} c_n$, where $c_n > 0$, converges. Determine, with justifications, whether the series

$$\sum_{n=1}^{\infty} \frac{2c_n+3}{c_n}$$

converges or diverges.

Since $\sum_{n=1}^{\infty} c_n$ converges, $\lim_{n\to\infty} c_n = 0$. Because

$$\lim_{n\to\infty} \frac{2c_n+3}{c_n} = \lim_{n\to\infty} \left(2 + \frac{3}{c_n}\right) = \infty,$$

the series diverges by the n^{th}-term test.

9 **3.** Find the open interval of convergence for the series

$$\sum_{n=2}^{\infty} n^2 a^{2n} (2x-1)^n,$$

where $a > 0$ is a constant. Determine whether the series converges at its right endpoint.

We write the series in the form $\displaystyle\sum_{n=2}^{\infty} n^2 a^{2n} 2^n \left(x - \frac{1}{2}\right)^n$. Its radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{n^2 a^{2n} 2^n}{(n+1)^2 a^{2n+2} 2^{n+1}} \right| = \frac{1}{2a^2}.$$

The open interval of convergence is therefore

$$\left| x - \frac{1}{2} \right| < \frac{1}{2a^2} \quad\Longrightarrow\quad -\frac{1}{2a^2} < x - \frac{1}{2} < \frac{1}{2a^2} \quad\Longrightarrow\quad \frac{1}{2} - \frac{1}{2a^2} < x < \frac{1}{2} + \frac{1}{2a^2}.$$

At the right endpoint $x = \dfrac{1}{2} + \dfrac{1}{2a^2}$, the series becomes

$$\sum_{n=2}^{\infty} n^2 a^{2n} 2^n \left(\frac{1}{2a^2}\right)^n = \sum_{n=2}^{\infty} n^2.$$

Since $\displaystyle\lim_{n\to\infty} n^2 = \infty$, the series diverges by the n^{th}-term test.

11 **4.** Determine whether the following series converge or diverge. Justify all conclusions. If a series converges, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1} \mathrm{Cos}^{-1}\left(\frac{1}{n}\right)$ (b) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n 3^{n+1}}{4^{2n-1}}$

(a) Since $\lim_{n\to\infty}\left[\frac{n}{n+1}\mathrm{Cos}^{-1}\left(\frac{1}{n}\right)\right] = \frac{\pi}{2}$, the series diverges by the n^{th}-term test.

(b) When we write the series in the form

$$\sum_{n=3}^{\infty} \frac{(-1)^n 3^{n+1}}{4^{2n-1}} = \sum_{n=3}^{\infty}\left[\frac{3}{4^{-1}}\left(-\frac{3}{16}\right)^n\right],$$

we see that it is geometric with common ratio $-3/16$. The series therefore converges, with sum

$$\frac{12(-3/16)^3}{1+3/16}.$$

11 **5.** Find the Taylor series about $x = 2$ for the function

$$f(x) = \frac{1}{(x+1)^2}.$$

You must use a method that guarantees that the series converges to the function. Express your answer in sigma notation, simplified as much as possible. For what values of x does the series converge to the function?

$\boxed{\text{x-2}}$

We begin with

$$\frac{1}{x+1} = \frac{1}{(x-2)+3} = \frac{1/3}{1+(x-2)/3} = \frac{1}{3}\sum_{n=0}^{\infty}\left[-\left(\frac{x-2}{3}\right)\right]^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-2)^n,$$

valid for $\left|-\left(\frac{x-2}{3}\right)\right| < 1 \quad \Longrightarrow \quad |x-2| < 3$. The radius of convergence of the series is 3. If we differentiate the series with respect to x,

$$-\frac{1}{(x+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}n(x-2)^{n-1}.$$

Thus,

$$\frac{1}{(x+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}n}{3^{n+1}}(x-2)^{n-1} = \sum_{n=-1}^{\infty}\frac{(-1)^{n+2}(n+1)}{3^{n+2}}(x-2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{3^{n+2}}(x-2)^n.$$

The series converges for $|x-2| < 3 \quad \Longrightarrow \quad -3 < x-2 < 3 \quad \Longrightarrow \quad -1 < x < 5.$