MATH2132 Test1 **February 28, 2017** **60 minutes**

Student Name - **Student Number -**

Values

6 **1.** If b is a constant, find the limit of the following sequence of functions, if it exists.

$$f_n(x) = \frac{n^3x^2 + nx + 4b}{3n^3x^2 - 2nx + 3b}, \qquad -1 \le x \le 10.$$

$$\lim_{n\to\infty} \frac{n^3x^2 + nx + 4b}{3n^3x^2 - 2nx + 3b} = \lim_{n\to\infty} \frac{x^2 + \frac{x}{n^2} + \frac{4b}{n^3}}{3x^2 - \frac{2x}{n^2} + \frac{3b}{n^3}} = \frac{x^2}{3x^2} = \frac{1}{3}, \quad x \neq 0$$

At $x = 0$ the sequence of functions becomes the sequence of constants $\{f_n(0)\} = \{4b/(3b)\} = \{4/3\}$, provided $b \neq 0$. This sequence has limit $4/3$. Thus,

$$\lim_{n\to\infty} \frac{n^3x^2 + nx + 4b}{3n^3x^2 - 2nx + 3b} = \begin{cases} 1/3, & -1 \le x \le 10, \quad x \neq 0 \\ 4/3, & x = 0,\ b \neq 0 \\ \text{Does not exist}, & b = 0. \end{cases}$$

3 **2.** Is it possible for a power series in x to converge for $-10 < x < 10$, converge for $x = 11$, and diverge for all other values of x? Explain.

No. Power series always converge on intervals. Since this is a power series in x, its open interval of convergence must be of the form $-R < x < R$. Since the series converges at $x = 11$, it follows that $R \ge 11$. The series must therefore converge for $10 \le x < 11$, contrary to the given.

8 **3.** Determine whether the following series converge or diverge. Justify all conclusions. If a series converges, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(1 + \frac{1}{n}\right)^n$ (b) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}}$

(a) Since $\left(1 + \dfrac{1}{n}\right)^n > 1$ for all $n \ge 1$, terms in the series are greater than $1/n$. Because the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges, so also does the given series.

(b) This is a geometric series with common ratio $-2/3$, which therefore converges to

$$\frac{\dfrac{(-1)^3 2^3}{3^4}}{1 + \dfrac{2}{3}} = -\frac{8}{81}\left(\frac{3}{5}\right) = -\frac{8}{135}.$$

11 **4.** Find the Taylor series about $x = 1$ for the function

$$f(x) = \frac{3x}{x+4}.$$

You must use a method that guarantees that the series converges to the function. Express your answer in sigma notation, simplified as much as possible. For what values of x does the series converge to the function?

$\boxed{\text{x-1}}$

$$\begin{aligned}
f(x) &= \frac{3x}{x+4} = 3 - \frac{12}{x+4} = 3 - \frac{12}{5+(x-1)} = 3 - \frac{12/5}{1+\frac{x-1}{5}} \\
&= 3 - \frac{12}{5}\sum_{n=0}^{\infty}\left(-\frac{x-1}{5}\right)^n, \qquad \left|\frac{x-1}{5}\right| < 1 \\
&= 3 - \frac{12}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{5^n}(x-1)^n, \qquad |x-1| < 5 \\
&= \left(3 - \frac{12}{5}\right) + \sum_{n=1}^{\infty}\frac{12(-1)^{n+1}}{5^{n+1}}(x-1)^n = \frac{3}{5} + \sum_{n=1}^{\infty}\frac{12(-1)^{n+1}}{5^{n+1}}(x-1)^n
\end{aligned}$$

The interval of convergence is

$$|x-1| < 5 \quad \Longrightarrow \quad -5 < x-1 < 5 \quad \Longrightarrow \quad -4 < x < 6.$$

11 **5.** Find the Maclaurin series for the function

$$f(x) = \sqrt[3]{x+c},$$

where c is a positive constant. You must use a method that guarantees that the series converges to the function. Express your answer in sigma notation, simplified as much as possible. What is the radius of convergence of the series?

$\boxed{\text{x}}$

$$\begin{aligned}
f(x) &= c^{1/3}\left(1+\frac{x}{c}\right)^{1/3} \\
&= c^{1/3}\left[1 + \frac{1}{3}\left(\frac{x}{c}\right) + \frac{(1/3)(-2/3)}{2!}\left(\frac{x}{c}\right)^2 + \frac{(1/3)(-2/3)(-5/3)}{3!}\left(\frac{x}{c}\right)^3 + \cdots\right] \\
&= c^{1/3}\left[1 + \frac{x}{3c} - \frac{2}{3^2c^2 2!}x^2 + \frac{(2)(5)}{3^3c^3 3!}x^3 + \cdots\right] \\
&= c^{1/3}\left[1 + \frac{x}{3c} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}[2\cdot 5\cdot 8\cdots(3n-4)]}{3^n c^n n!}x^n\right] \\
&= c^{1/3} + \frac{x}{3c^{2/3}} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}[2\cdot 5\cdot 8\cdots(3n-4)]}{3^n c^{n-1/3} n!}x^n
\end{aligned}$$

Since the expansion is valid for $|x/c| < 1$, which implies that $|x| < c$, the radius of convergence is $R = c$.