MATH2132 Test1 **February 6, 2018** **60 minutes**

Student Name - **Student Number -**

Values

6 **1.** Find the limit of the following sequence of functions on the interval $0 \leq x < \infty$, if it exists.

$$f_n(x) = \frac{2^n x^3 - nx^2 + 4}{2^{n+1}x^2 + 3x - 11}.$$

$$\lim_{n\to\infty} \frac{2^n x^3 - nx^2 + 4}{2^{n+1}x^2 + 3x - 11} = \lim_{n\to\infty} \frac{x^3 - \dfrac{nx^2}{2^n} + \dfrac{4}{2^n}}{2x^3 + \dfrac{3x}{2^n} - \dfrac{11}{2^n}} = \frac{x^3}{2x^2} = \frac{x}{2},$$

provided that $x \neq 0$. When $x = 0$ $\{f_n(0)\} = \{-4/11\}$, which has limit $-4/11$. Thus

$$\lim_{n\to\infty} \{f_n(x)\} = \begin{cases} x/2, & x > 0 \\ -4/11, & x = 0. \end{cases}$$

7 **2.** Find all values of the constant b for which the series of constants

$$\sum_{n=2}^{\infty} \frac{(-1)^n b^n}{3^{n+1}}$$

converges. For these values of b, determine the sum of the series.

This is a geometric series with common ratio $-b/3$. It therefore converges when

$$\left| -\frac{b}{3} \right| < 1 \qquad \Longrightarrow \qquad |b| < 3.$$

For these values of b, the sum of the series is

$$\frac{\frac{(-1)^2 b^2}{3^3}}{1 + b/3} = \frac{b^2}{9(b+3)}.$$

10 **3.** (a) Find the open interval of convergence of the power series

$$\sum_{n=3}^{\infty} \frac{n}{(2n+1)5^n}(x-3)^{2n+1}.$$

(b) Does the series converge at the left end point of the open interval of convergence? Justify your answer.

(a) We begin by writing the series in the form

$$(x-3)\sum_{n=3}^{\infty} \frac{n}{(2n+1)5^n}(x-3)^{2n}.$$

We now set $y=(x-3)^2$, in which case the series (less the term outside) becomes

$$\sum_{n=1}^{\infty} \frac{n}{(2n+1)5^n}y^n.$$

The radius of convergence of this series is

$$R_y = \lim_{n\to\infty}\left|\frac{\dfrac{n}{(2n+1)5^n}}{\dfrac{n+1}{(2n+3)5^{n+1}}}\right| = 5.$$

Thus, the radius of convergence of the given series is $R_x=\sqrt{5}$. Its open interval of convergence is

$$-\sqrt{5} < x-3 < \sqrt{5} \qquad\Longrightarrow\qquad 3-\sqrt{5} < x < 3+\sqrt{5}.$$

(b) At $x = 3-\sqrt{5}$, the series becomes

$$\sum_{n=3}^{\infty} \frac{n}{(2n+1)5^n}(3-\sqrt{5}-3)^{2n+1} = \sum_{n=3}^{\infty}\frac{-\sqrt{5}n}{2n+1}.$$

Since $\lim_{n\to\infty}\dfrac{-\sqrt{5}n}{2n+1} = -\dfrac{\sqrt{5}}{2} \neq 0$, the series diverges by the n^{th}-term test.

4. It is known that the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence $\overline{R}$, where $0 < \overline{R} < \infty$. Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n^2 a_n}{2^n} (x-c)^n.$$

The radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 a_n}{2^n}(x-c)^n$ is

$$R = \lim_{n\to\infty} \left| \frac{\dfrac{n^2 a_n}{2^n}}{\dfrac{(n+1)^2 a_{n+1}}{2^{n+1}}} \right| = 2 \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = 2\overline{R}.$$

13 **5.** (a) Find the Taylor series about $x = -2$ for the function

$$f(x) = \frac{x+2}{(4+x)^2}.$$

You must use a method that guarantees that the series converges to the function. Express your answer in sigma notation, simplified as much as possible.

(b) What is the radius of convergence of the series?

(a) $\boxed{\text{x+2}}$

$$\frac{1}{4+x} = \frac{1}{(x+2)+2} = \frac{1/2}{1+\dfrac{x+2}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x+2}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x+2)^n,$$

valid for

$$\left|-\frac{x+2}{2}\right| < 1 \quad\Longrightarrow\quad |x+2| < 2 \quad\Longrightarrow\quad -2 < x+2 < 2 \quad\Longrightarrow\quad -4 < x < 0.$$

Differentiation gives

$$-\frac{1}{(4+x)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n n}{2^{n+1}}(x+2)^{n-1}.$$

Thus,

$$\frac{x+2}{(4+x)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}n}{2^{n+1}}(x+2)^n = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{2^{n+1}}(x+2)^n.$$

(b) The radius of convergence of the series is $R = 2$.