MATH 2132 Test 2 **November 3, 2011** **60 minutes**

Student Name - **Student Number -**

Values

15 **1.** (a) Find the Taylor series about $x = 2$ for the function $\dfrac{1}{(3+2x)^{1/3}}$. Express your answer in sigma notation simplified as much as possible. You must use a technique that ensures that the series converges to the function.

(b) What is the radius of convergence of the series?

(a)

$$\frac{1}{(3+2x)^{1/3}} = \frac{1}{[7+2(x-2)]^{1/3}} = \frac{1}{7^{1/3}}\left[1+\frac{2}{7}(x-2)\right]^{-1/3}$$

$$= 7^{-1/3}\left\{1+\left(-\frac{1}{3}\right)\left[\frac{2}{7}(x-2)\right]+\frac{(-1/3)(-4/3)}{2!}\left[\frac{2}{7}(x-2)\right]^2 + \frac{(-1/3)(-4/3)(-7/3)}{3!}\left[\frac{2}{7}(x-2)\right]^3+\cdots\right\}$$

$$= 7^{-1/3}\left[1-\frac{2}{3\cdot 7}(x-2)+\frac{4\cdot 2^2}{3^2 7^2 2!}(x-2)^2-\frac{(4\cdot 7)2^3}{3^3 7^3 3!}(x-2)^3+\cdots\right]$$

$$= 7^{-1/3}\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n 2^n[1\cdot 4\cdot 7\cdots(3n-2)]}{3^n 7^n n!}(x-2)^n\right\}$$

(b) The expansion is valid for $\left|\frac{2}{7}(x-2)\right| < 1 \implies |x-2| < \frac{7}{2}$. The radius of convergence is $R = 7/2$.

13 **2.** (a) Find the Maclaurin series for the function $f(x) = \dfrac{x^4}{(4-3x)^2}$. Express your final answer in sigma notation simplified as much as possible.

(b) What is the interval of convergence of the series?

(a) $\dfrac{1}{4-3x} = \dfrac{1}{4(1-3x/4)} = \dfrac{1}{4}\sum_{n=0}^{\infty}\left(\dfrac{3x}{4}\right)^n = \sum_{n=0}^{\infty}\dfrac{3^n}{4^{n+1}}x^n$,
valid for $|3x/4| < 1$, or $|x| < 4/3$. Because the radius of convergence is positive, we may differentiate this series term-by-term,

$$\frac{3}{(4-3x)^2} = \sum_{n=0}^{\infty}\frac{3^n n}{4^{n+1}}x^{n-1}.$$

Thus,

$$\frac{x^4}{(4-3x)^2} = \sum_{n=0}^{\infty}\frac{3^{n-1}n}{4^{n+1}}x^{n+3} = \sum_{n=3}^{\infty}\frac{3^{n-4}(n-3)}{4^{n-2}}x^n = \sum_{n=4}^{\infty}\frac{3^{n-4}(n-3)}{4^{n-2}}x^n.$$

(b) Since end points of an open interval of convergence are never picked up under differentiation, the interval of convergence is $|x| < 4/3$.

12 **3.** Evaluate

$$\sum_{n=2}^{\infty} \frac{2^n}{(n+1)!} x^n.$$

Justify all steps in your solution.

Method 1: The radius of convergence of the series is $R = \lim_{n\to\infty} \left| \frac{2^n/(n+1)!}{2^{n+1}/(n+2)!} \right| = \infty$. If we set $S(x) = \sum_{n=2}^{\infty} \frac{2^n}{(n+1)!} x^n$, then

$$\begin{aligned} xS(x) &= \sum_{n=2}^{\infty} \frac{2^n}{(n+1)!} x^{n+1} = \sum_{n=3}^{\infty} \frac{2^{n-1}}{n!} x^n = \frac{1}{2} \sum_{n=3}^{\infty} \frac{(2x)^n}{n!} \\ &= \frac{1}{2}[e^{2x} - 1 - 2x - 2x^2] = \frac{1}{2}(e^{2x} - 1) - x - x^2. \end{aligned}$$

Thus, $S(x) = \frac{1}{2x}(e^{2x} - 1) - 1 - x$, provided $x \neq 0$. The sum at $x = 0$ is 0.

Method 2: The radius of convergence of the series is $R = \lim_{n\to\infty} \left| \frac{2^n/(n+1)!}{2^{n+1}/(n+2)!} \right| = \infty$. If we set $S(x) = \sum_{n=2}^{\infty} \frac{2^n}{(n+1)!} x^n$, then

$$xS(x) = \sum_{n=2}^{\infty} \frac{2^n}{(n+1)!} x^{n+1}.$$

Differentiation gives

$$\frac{d}{dx}[xS(x)] = \sum_{n=2}^{\infty} \frac{2^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} - 1 - 2x = e^{2x} - 1 - 2x.$$

Integration now gives

$$xS(x) = \frac{1}{2} e^{2x} - x - x^2 + C.$$

Substitution of $x = 0$ yields $0 = 1/2 + C$, and therefore

$$xS(x) = \frac{1}{2} e^{2x} - x - x^2 - \frac{1}{2} \quad \Longrightarrow \quad S(x) = \frac{1}{2x}(e^{2x} - 1) - 1 - x,$$

valid for all x except $x = 0$. The sum at $x = 0$ is 0.

10 **4.** Find, in explicit form $y = f(x)$, the solution of the initial value problem

$$x^2 y \frac{dy}{dx} + x^2 = 1, \qquad y(1) = 1.$$

The differential equation is separable,

$$y\,dy = \frac{1 - x^2}{x^2}\,dx.$$

A 1-parameter family of solutions is defined implicitly by

$$\int y\,dy = \int \left(\frac{1}{x^2} - 1\right) dx \qquad \Longrightarrow \qquad \frac{y^2}{2} = -\frac{1}{x} - x + C.$$

For $y(1) = 1$,

$$\frac{1}{2} = -1 - 1 + C \qquad \Longrightarrow \qquad C = \frac{5}{2}.$$

Thus,

$$y^2 = -\frac{2}{x} - 2x + 5 \qquad \Longrightarrow \qquad y = \pm\sqrt{5 - \frac{2}{x} - 2x}.$$

But only $y = \sqrt{5 - \frac{2}{x} - 2x}$ satisfies the initial condition.