MATH2132 Test 2 **October 24, 2013** **60 minutes**

Student Name - **Student Number -**

Values

15 **1.** Find the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}n}(x+1)^{3n+1}.$$

Justify all statements.

We write

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}n}(x+1)^{3n+1} = (x+1)\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}n}(x+1)^{3n},$$

and set $y = (x+1)^3$. The series is then

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}n} y^n.$$

The radius of convergence of this series is

$$R_y = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^n}{2^{n+1}n}}{\dfrac{(-1)^{n+1}}{2^{n+2}(n+1)}} \right| = 2.$$

The radius of convergence of the original series is therefore $R_x = 2^{1/3}$. The open interval of convergence is

$$|x+1| < 2^{1/3} \quad \Longrightarrow \quad -2^{1/3} - 1 < x < 2^{1/3} - 1.$$

At $x = -2^{1/3} - 1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}n}(-2^{1/3})^{3n+1} = \frac{-1}{2^{2/3}} \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is the harmonic series which diverges.

At $x = 2^{1/3} - 1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}n}(2^{1/3})^{3n+1} = \frac{1}{2^{2/3}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This is the alternating harmonic series which converges. The interval of convergence is therefore $-2^{1/3} - 1 < x \le 2^{1/3} - 1$.

15 **2.** Find, if possible, the Maclaurin series for the function

$$f(x) = \frac{x+5}{(4-x)^2}.$$

If the series exists, derive it with a method that guarantees that the series converges to the function. Express your answer in sigma notation, simplified as much as possible. What is the interval of convergence of the series? If the series does not exist, explain why not.

The box is $\boxed{x}$. We begin with

$$\frac{1}{4-x} = \frac{1}{4\left(1-\dfrac{x}{4}\right)} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}}, \qquad \left|\frac{x}{4}\right| < 1 \implies -4 < x < 4.$$

Differentiation with respect to x gives

$$\frac{1}{(4-x)^2} = \sum_{n=0}^{\infty}\frac{n}{4^{n+1}}x^{n-1}.$$

We now multiply by $x+5$,

$$\begin{aligned}
\frac{x+5}{(4-x)^2} &= (x+5)\sum_{n=0}^{\infty}\frac{n}{4^{n+1}}x^{n-1} = \sum_{n=0}^{\infty}\frac{n}{4^{n+1}}x^n + \sum_{n=0}^{\infty}\frac{5n}{4^{n+1}}x^{n-1} \\
&= \sum_{n=0}^{\infty}\frac{n}{4^{n+1}}x^n + \sum_{n=-1}^{\infty}\frac{5(n+1)}{4^{n+2}}x^n = \sum_{n=0}^{\infty}\left[\frac{n}{4^{n+1}} + \frac{5(n+1)}{4^{n+2}}\right]x^n \\
&= \sum_{n=0}^{\infty}\left[\frac{4n+5(n+1)}{4^{n+2}}\right]x^n = \sum_{n=0}^{\infty}\left(\frac{9n+5}{4^{n+2}}\right)x^n.
\end{aligned}$$

Since differentiation of a series never picks up an end point of an open interval of convergence, the interval of convergence is $-4 < x < 4$.

5 **3.** Find, if possible, the Taylor series for the function

$$f(x) = (x-5)\ln(4-2x)$$

about $x = 5$. If the series exists, derive it with a method that guarantees that the series converges to the function. Express your answer in sigma notation, simplified as much as possible. What is the radius of convergence of the series? If the series does not exist, explain why not.

The first term in a Taylor series is the value of the function at the point of expansion. In this case, it is

$$f(5) = (0)\ln(-6).$$

Since this is undefined, there is no Taylor series.

15 **4.** Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(n+1)}{n!} x^{n-1}$.

If we write $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{(n+1)!} x^n$, then the radius of convergence of the series is

$$R = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(n+2)}{(n+1)!}}{\dfrac{(-1)^{n+2}(n+3)}{(n+2)!}} \right| = \infty.$$

Let us set $S(x) = \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(n+1)}{n!} x^{n-1}$. Then

$$x\,S(x) = \sum_{n=2}^{\infty} \frac{(-1)^n(n+1)}{n!} x^n.$$

Integration gives

$$\begin{aligned}\int x\,S(x)\,dx &= \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} x^{n+1} + C = x \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} x^n + C \\ &= x \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n - 1 + x \right] + C = x(e^{-x} - 1 + x) + C \\ &= xe^{-x} - x + x^2 + C.\end{aligned}$$

Differentiation now gives

$$x\,S(x) = -xe^{-x} + e^{-x} - 1 + 2x,$$

and therefore

$$S(x) = \frac{e^{-x}(1-x) + 2x - 1}{x}, \qquad \text{provided } x \neq 0.$$

When $x = 0$, the sum of the series is $S(0) = 0$. Thus,

$$\sum_{n=2}^{\infty} \frac{(-1)^n(n+1)}{n!} x^{n-1} = \begin{cases} \dfrac{e^{-x}(1-x) + 2x - 1}{x}, & -\infty < x < \infty,\ x \neq 0 \\ 0, & x = 0. \end{cases}$$