MATH2132 Test 2 **November 1, 2018** **60 minutes**

Student Name - **Student Number -**

Values

16 **1.** Find the Taylor series about $x = 1$ for the function $f(x) = 1/(3x+2)^{3/2}$. You must use a method that guarantees that the series converges to $f(x)$. Express your answer in sigma notation, simplified as much as possible. What is the radius of convergence of the series?

$\boxed{\text{x-1}}$

$$\frac{1}{(3x+2)^{3/2}} = \frac{1}{[3(x-1)+5]^{3/2}} = \frac{1}{5\sqrt{5}[1+3(x-1)/5]^{3/2}} = \frac{1}{5\sqrt{5}}\left[1+\frac{3}{5}(x-1)\right]^{-3/2}$$

$$= \frac{1}{5\sqrt{5}}\left[1+\left(-\frac{3}{2}\right)\left(\frac{3}{5}\right)(x-1)+\frac{(-3/2)(-5/2)}{2!}\left(\frac{3}{5}\right)^2(x-1)^2+\cdots\right]$$

$$= \frac{1}{5\sqrt{5}}\left[1-\frac{3\cdot 3}{2\cdot 5}(x-1)+\frac{3^2(3\cdot 5)}{2^2 5^2 2!}(x-1)^2-\frac{3^3(3\cdot 5\cdot 7)}{2^3 5^3 3!}(x-1)^3+\cdots\right]$$

$$= \frac{1}{5\sqrt{5}}\sum_{n=0}^{\infty}\frac{(-1)^n 3^n[1\cdot 3\cdot 5\cdots(2n+1)]}{2^n 5^n n!}(x-1)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 3^n (2n+1)!}{2^{2n}5^{n+3/2}(n!)^2}(x-1)^n.$$

Since the series converges for

$$\left|\frac{3}{5}(x-1)\right| < 1 \qquad \Longrightarrow \qquad |x-1| < \frac{5}{3},$$

the radius of convergence is $5/3$.

8 **2.** Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{a^{2n+2}(2n+1)!} x^{2n+2},$$

where a is a constant. For what values of x is your sum valid?

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n}{a^{2n+2}(2n+1)!} x^{2n+2} &= \frac{x}{a} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{x}{a}\right)^{2n+1} = \frac{x}{a}\left[\sin\left(\frac{x}{a}\right) - \frac{x}{a}\right] \\ &= \frac{x}{a} \sin\left(\frac{x}{a}\right) - \frac{x^2}{a^2}.\end{aligned}$$

This is valid for all real x.

10 **3.** Find a 1-parameter family of solutions to the differential equation

$$\frac{dy}{dx} = \frac{x^4 e^{2x} + 2y}{x}.$$

Does your family contain any singular solutions? If it does, find one; if it does not, explain why not.

When we write the differential equation in the form

$$\frac{dy}{dx} - \frac{2}{x}y = x^3 e^{2x},$$

we see that it is linear first-order. An integrating factor is

$$e^{\int (-2/x)dx} = e^{-2\ln|x|} = \frac{1}{x^2},$$

provided $x \neq 0$. We multiply each term in the differential equation by $1/x^2$ to get

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = xe^{2x} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{y}{x^2}\right) = xe^{2x}.$$

Integration gives

$$\frac{y}{x^2} = \int xe^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C.$$

A 1-parameter family of solutions is therefore

$$y(x) = \frac{x^3}{2}e^{2x} - \frac{x^2}{4}e^{2x} + Cx^2.$$

Since the differential equation is linear, this is a general solution, and there can be no singular solutions.

6 **4.** Three chemicals A, B, and C combine to form chemical D in such a way that 1 unit of A, 2 units of B, and 2 units of C create 5 units of D. The rate at which D is formed is proportional to the product of the amount of A and the square of the amount of C in the mixture, but it does not depend on the amount of B. If 10 units of A, 12 units of B, and 15 units of C are placed in a container at time $t = 0$, find an initial-value problem for the amount of D given that 3 units of D were in the container at that time.

If we let $x(t)$ represent the number of units of D in the mixture at time t, then

$$\frac{dx}{dt} = k\left[10 - \frac{x-3}{5}\right]\left[15 - \frac{2(x-3)}{5}\right]^2, \qquad x(0) = 3.$$