MATH2132 Test2 **November 5, 2020** **60 minutes**

Values

7 **1.** (a) Find a 1-parameter family of solutions for the differential equation

$$\frac{1}{x}\frac{dy}{dx} = 1 - \frac{4y}{x^2+1}.$$

(b) Is your family a general solution? Explain.

(a) When we write the differential equation in the form,

$$\frac{dy}{dx} + \frac{4xy}{x^2+1} = x,$$

we see that it is linear with integrating factor

$$e^{\int 4x/(x^2+1)dx} = e^{2\ln(x^2+1)} = (x^2+1)^2.$$

Multiplication of the differential equation by this gives

$$(x^2+1)^2\frac{dy}{dx} + 4x(x^2+1)y = x(x^2+1)^2 \quad\Longrightarrow\quad \frac{d}{dx}[(x^2+1)^2y] = x(x^2+1)^2.$$

Integration now gives

$$(x^2+1)^2y = \frac{1}{6}(x^2+1)^3 + C \quad\Longrightarrow\quad y(x) = \frac{1}{6}(x^2+1) + \frac{C}{(x^2+1)^2}.$$

(b) Because we have a 1-parameter family of solutions to a linear, first-order differential equation, the solution is general.

7 **2.** Find a general solution for the differential equation

$$\frac{d^3y}{dx^3} + 2\frac{dy}{dx} - 3y = 3\sin 2x.$$

The auxiliary equation is

$$0 = m^3 + 2m - 3 = (m-1)(m^2 + m + 3),$$

with solutions $m = 1$ and $m = (-1 \pm \sqrt{1-12})/2 = (-1 \pm \sqrt{11}i)/2$. A general solution of the associated homogeneous equation is

$$y_h(x) = C_1e^x + e^{-x/2}\left(C_2 \cos\frac{\sqrt{11}x}{2} + C_3 \sin\frac{\sqrt{11}x}{2}\right).$$

If we substitute a particular solution $y_p(x) = A\sin 2x + B\cos 2x$ into the nonhomogeneous equation, we get

$$(-8A\cos 2x + 8B\sin 2x) + 2(2A\cos 2x - 2B\sin 2x) - 3(A\sin 2x + B\cos 2x) = 3\sin 2x.$$

When we equate coefficients of $\sin 2x$ and $\cos 2x$, we get

$$8B - 4B - 3A = 3, \quad -8A + 4A - 3B = 0, \qquad \Longrightarrow \qquad A = -\frac{9}{25}, \quad B = \frac{12}{25}.$$

Thus, $y_p(x) = -\frac{9}{25}\sin 2x + \frac{12}{25}\cos 2x$, and a general solution of the nonhomogeneous equation is

$$y(x) = C_1e^x + e^{-x/2}\left(C_2 \cos\frac{\sqrt{11}x}{2} + C_3 \sin\frac{\sqrt{11}x}{2}\right) - \frac{9}{25}\sin 2x + \frac{12}{25}\cos 2x.$$

8 **3.** (a) Solve the following initial-value problem

$$2\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 18y = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

(b) Find the maximum value for $y(x)$.

(a) The auxiliary equation is $0 = m^2 + 10m + 9 = (m+1)(m+9)$ with solutions $m = -1$ and $m = -9$. A general solution of the differential equation is

$$y(x) = C_1 e^{-x} + C_2 e^{-9x}.$$

The initial conditions require

$$0 = y(0) = C_1 + C_2, \quad 1 = y'(0) = -C_1 - 9C_2 \qquad \Longrightarrow \qquad C_1 = \frac{1}{8}, \quad C_2 = -\frac{1}{8}.$$

Thus,

$$y(x) = \frac{1}{8}(e^{-x} - e^{-9x}).$$

(b) $y(x)$ is a maximum when

$$0 = y'(x) = \frac{1}{8}(-e^{-x} + 9e^{-9x}) \quad \Longrightarrow \quad e^{8x} = 9 \quad \Longrightarrow \quad x = \frac{1}{8}\ln 9.$$

The maximum value of $y(x)$ is therefore

$$\frac{1}{8}(e^{-(1/8)\ln 9} - e^{(-9/8)\ln 9}).$$

8 **4.** Solve the following initial-value problem

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}\left(1 + 2\frac{dy}{dx}\right), \qquad y(0) = 1, \quad y'(0) = -1/2.$$

Since both x and y are missing, there are two solutions.

Solution 1: Because y is explicitly missing, we set $v = \dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$,

$$\frac{dv}{dx} = (1+2v)v \qquad \Longrightarrow \qquad \frac{dv}{v(1+2v)} = dx, \qquad \text{provided } v \neq 0, \quad v \neq -1/2.$$

A one-parameter family of solutions is defined implicitly by

$$\int \left(\frac{1}{v} - \frac{2}{1+2v}\right) dv = x + C \qquad \ln|v| - \ln|1+2v| = x + C.$$

Thus,

$$\ln\left|\frac{v}{1+2v}\right| = x + C \qquad \Longrightarrow \qquad \frac{v}{1+2v} = De^x.$$

When we multiply by $1 + 2v$,

$$v = (1+2v)De^x \qquad \Longrightarrow \qquad v(x) = \frac{De^x}{1 - 2De^x}.$$

Since $v = -1/2$ when $x = 0$,

$$-\frac{1}{2} = \frac{D}{1-2D} \qquad -1 + 2D = 2D, \qquad \text{an impossibility.}$$

We must therefore have $v = 0$ or $v = -1/2$. The first is not possible since $y'(0) = -1/2$. Thus

$$v = \frac{dy}{dx} = -\frac{1}{2} \qquad \Longrightarrow \qquad y(x) = -\frac{x}{2} + E.$$

The initial condition $y(0) = 1$ requires $E = 1$, and the solution is $y(x) = 1 - x/2$.

Solution 2: Since x is explicitly missing we set $v = \dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2} = v\dfrac{dv}{dy}$,

$$v\frac{dv}{dy} = (1+2v)v \qquad \Longrightarrow \qquad \frac{dv}{1+2v} = dy, \quad \text{provided } v \neq 0, \quad v \neq -1/2.$$

A one-paraneter family of solutions is defined implicitly by

$$\frac{1}{2}\ln|1+2v| = y + C \qquad \Longrightarrow \qquad \ln|1+2v| = 2y + 2C.$$

Exponentiation gives

$$|1+2v| = e^{2y+2C} \qquad \Longrightarrow \qquad 1 + 2v = De^{2y} \qquad \Longrightarrow \qquad v = -\frac{1}{2} + \frac{D}{2}e^{2y}.$$

Since $v = -1/2$ whe $y = 1$,

$$-\frac{1}{2} = -\frac{1}{2} + \frac{D}{2}e^2.$$

This gives $D = 0$, which is not possible since $D = \pm e^{2C}$. We must therefore have $v = 0$ or $v = -1/2$. The first is not possible since $y'(0) = -1/2$. Thus

$$v = \frac{dy}{dx} = -\frac{1}{2} \qquad \Longrightarrow \qquad y(x) = -\frac{x}{2} + E.$$

The initial condition $y(0) = 1$ requires $E = 1$, and the solution is $y(x) = 1 - x/2$.