MATH2132 Test 2 Solutions

Values

10 **1.** (a) Find a one-parameter family of solutions, defined explicitly, for the differential equation

$$\frac{dy}{dx} = -2x(1+y)^2.$$

(b) Are there any singular solutions of your family?

(c) Find the solution of the differential equation that also satisfies the condition $y(1) = -1$.

(a) The equation is separable,

$$\frac{dy}{(1+y)^2} = -2x\,dx, \qquad (y \neq -1).$$

A 1-parameter family of solutions is defined implicitly by

$$\frac{-1}{1+y} = -x^2 + C \qquad \Longrightarrow \qquad 1 + y = \frac{1}{x^2 - C}.$$

Explicit solutions are

$$y(x) = \frac{1}{x^2 - C} - 1.$$

(b) $y(x) = -1$ is a solution of the differential equation, and because it is not in the family of solutions, it is a singular solution of the family.

(c) If we impose the initial condition,

$$-1 = \frac{1}{1 - C} - 1,$$

which has no solution for C. But, notice that the singular solution $y(x) = -1$ satisfies the initial condition. Hence, $y(x) = -1$ is the solution of the initial-value problem.

5 **2.** You are given that the roots of the auxiliary equation associated with the differential equation

$$\phi(D)y = x^2 - 3x + e^{3x} + 4\sin x$$

are $m = 0, 0, \pm 3, -\sqrt{7}, 1 \pm i$. Determine the form of a particular solution of the differential equation as predicted by undetermined coefficients.

Since

$$y_h(x) = (C_1 + C_2x) + C_3e^{3x} + C_4e^{-3x} + C_5e^{-\sqrt{7}x} + e^x(C_6\cos x + C_7\sin x),$$

the form of a particular solution is

$$y_p(x) = (Ax^4 + Bx^3 + Cx^2) + Dxe^{3x} + E\sin x + F\cos x.$$

8 **3.** (a) Find two solutions of the differential equation

$$2x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} - 4y = 0$$

of the form $y(x) = x^m$, where m is a constant.

(b) Use the solutions in part (a) to find a 2-parameter family of solutions of the differential equation.

(c) Can you claim that your solution in part (b) is a general solution of the differential equation? Explain.

(a) Since $y(x) = x^m$ is to be a solution of the differential equation, we may substitute it into the equation,

$$0 = 2x^2m(m-1)x^{m-2} - 5xmx^{m-1} - 4x^m = x^m(2m^2 - 7m - 4) = x^m(2m+1)(m-4).$$

Thus, $m = 4$ and $m = -1/2$. Solutions of the differential equation are therefore $y_1(x) = x^4$ and $y_2(x) = 1/\sqrt{x}$.

(b) Because the differential equation is linear and homogeneous, we may superpose solutions and a 2-parameter family of solutions is

$$y(x) = C_1x^4 + \frac{C_2}{\sqrt{x}}.$$

(c) Because the differential equation is linear (and homogeneous), a 2-parameter family of solutions is a general solution.

11 **4.** A 1-kilogram mass is suspended from a spring with constant $k = 32$ newtons per metre. It is raised 10 centimetres above its equilibrium position and released. During its subsequent motion, it is subject to damping that is 12 times its speed. Find the time(s), if any, when the mass is 1 centimetre above its equilibrium position.

The initial-value problem for displacement of the mass is

$$(1)\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 32x = 0, \qquad x(0) = \frac{1}{10}, \quad x'(0) = 0.$$

The auxiliary equation $0 = m^2 + 12m + 32 = (m+4)(m+8)$ has roots $m = -4$ and $m = -8$. A general solution of the differential equation is

$$x(t) = C_1 e^{-4t} + C_2 e^{-8t}.$$

The initial conditions require

$$\frac{1}{10} = x(0) = C_1 + C_2, \qquad 0 = x'(0) = -4C_1 - 8C_2.$$

These imply that $C_1 = 1/5$ and $C_2 = -1/10$. Thus,

$$x(t) = \frac{1}{5}e^{-4t} - \frac{1}{10}e^{-8t}.$$

The mass is 1 centimetre above its equilibrium if, and when,

$$\frac{1}{100} = \frac{1}{5}e^{-4t} - \frac{1}{10}e^{-8t}.$$

If we multiply by $100e^{8t}$,

$$e^{8t} = 20e^{4t} - 10 \qquad \Longrightarrow \qquad e^{8t} - 20e^{4t} + 10 = 0.$$

This is a quadratic equation in e^{4t}, and therefore

$$e^{4t} = \frac{20 \pm \sqrt{400 - 40}}{2} = 10 \pm 3\sqrt{10}.$$

Since e^{4t} must be positive,

$$e^{4t} = 10 + 3\sqrt{10} \quad \Longrightarrow \quad t = \frac{1}{4}\ln(10 + 3\sqrt{10}).$$

6 **5.** A certain chemical dissolves in water at a rate proportional to the product of the amount of undissolved chemical and the difference between concentrations in a saturated solution and the existing concentration in the solution. A saturated solution contains 25 grams of chemical in 100 millilitres of solution. If 50 grams of chemical is added to 100 millilitres of water, set up, but do **NOT** solve, an initial-value problem for the amount of dissolved chemical in the water as a function of time.

If $C(t)$ represents the number of grams of dissolved chemical, then the initial-value problem for $C(t)$ is

$$\frac{dC}{dt} = k(50 - C)\left(\frac{25}{100} - \frac{C}{100}\right), \qquad C(0) = 0.$$