MATH 2132 Test 2 Solutions

1. (a) Find a 1-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = y^2 - 4.$$

(b) Find the solution of the differential equation that also satisfies the condition $y(0) = -2$.

(a) We can separate the differential equation,

$$\frac{dy}{y^2 - 4} = dx, \quad \text{provided } y \neq \pm 2.$$

A 1-parameter family of solutions is defined implicitly by

$$\begin{aligned}
\int \left(\frac{1/4}{y-2} - \frac{1/4}{y+2} \right) dy &= x + C \\
\frac{1}{4} \ln|y-2| - \frac{1}{4} \ln|y+2| &= x + C \\
\ln\left|\frac{y-2}{y+2}\right| &= 4x + 4C \\
\left|\frac{y-2}{y+2}\right| &= e^{4x+4C} \\
\frac{y-2}{y+2} &= \pm e^{4C} e^{4x} = De^{4x}, \quad \text{where } D = \pm e^{4C} \\
y - 2 &= (y+2)De^{4x} \\
y &= \frac{2De^{4x} + 2}{1 - De^{4x}}
\end{aligned}$$

(b) For the solution to satisfy $y(0) = -2$, we set

$$-2 = \frac{2D + 2}{1 - D} \qquad \Longrightarrow \qquad -2 + 2D = 2D + 2$$

an impossibility. But notice that the function $y(x) = -2$ satisfies the differential equation and the initial condition. Hence, the solution of the initial-value problem is $y(x) = -2$.

2. (a) A 2000 litre tank contains 1000 litres of water in which 5 kilograms of salt have been dissolved. A brine mixture with concentration $2\sin t$ kilograms of salt for each 100 litres of water is added to the tank at 10 millilitres per second. At the same time, mixture is being drawn from the bottom of the tank at 5 millilitres per second. Assume that the mixture in the tank is always sufficiently well-stirred that at any given time, concentration of salt is the same at all points in the tank. Set up, but do **NOT** solve, an initial-value problem for the number of grams of salt in the tank as a function of time (in seconds).

(b) For how long is the differential equation in part (a) valid?

(a) If $S(t)$ represents the number of grams of salt in the tank, then the initial-value problem for $S(t)$ is

$$\frac{dS}{dt} = \frac{1}{5}\sin t - \frac{5S}{10^6 + 5t}, \qquad S(0) = 5000.$$

(b) The differential equation is valid until the tank fills. This occurs when

$$10^6 + 5t = 2(10^6) \qquad \Longrightarrow \qquad t = 200\,000.$$

3. Find a general solution for the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 6y = 3x^2 - 2e^{5x}.$$

The auxiliary equation is $m^2 + 4m + 6 = 0$ with roots $m = (-4 \pm \sqrt{16-24})/2 = -2 \pm \sqrt{2}i$. A general solution of the associated homogeneous equation is $y_h(x) = e^{-2x}(C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x)$. A particular solution is of the form $y_p(x) = Ax^2 + Bx + C + De^{5x}$. Substitution into the differential equation gives

$$(2A + 25De^{5x}) + 4(2Ax + B + 5De^{5x}) + 6(Ax^2 + Bx + C + De^{5x}) = 3x^2 - 2e^{5x}.$$

When we equate coefficients of x^2, x, 1, and e^{5x}, we obtain

$$6A = 3, \quad 8A + 6B = 0, \quad 2A + 4B + 6C = 0, \quad 51D = -2.$$

The solution of these is $A = 1/2$, $B = -2/3$, $C = 5/18$, and $D = -2/51$. Thus, $y_p(x) = x^2/2 - 2x/3 + 5/18 - (2/51)e^{5x}$, and a general solution of the given differential equation is

$$y(x) = e^{-2x}(C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x) + \frac{x^2}{2} - \frac{2x}{3} + \frac{5}{18} - \frac{2}{51}e^{5x}.$$

4. Find the form of a particular solution of the following differential equation as predicted by undetermine coefficients. Do **NOT** attempt to evaluate the coefficients.

$$\frac{d^5y}{dx^5} + 8\frac{d^3y}{dx^3} + 16\frac{dy}{dx} = x^2 + 2\sin 2x - e^{3x}$$

The auxiliary equation is

$$0 = m^5 + 8m^3 + 16m = m(m^2 + 4)^2$$

with roots $m = 0, \pm 2i, \pm 2i$. A general solution of the associated homogeneous equation is

$$y_h(x) = C_1 + (C_2 + C_3x)\cos 2x + (C_4 + C_5x)\sin 2x.$$

A particular solution takes the form

$$y_p(x) = Ax^3 + Bx^2 + Cx + Dx^2\sin 2x + Ex^2\cos 2x + He^{3x}.$$