MATH 2132 Test 2 **March 8, 2016** **75 minutes**

Student Name - **Student Number -**

Values

10 **1.** Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{(2n)!} x^{2n}.$$

If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{(2n)!} y^n$. Its radius of convergence is

$$R_y = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^n(n+2)}{(2n)!}}{\dfrac{(-1)^{n+1}(n+3)}{(2n+2)!}} \right| = \lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!}{(2n)!} = \infty.$$

We set

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{(2n)!} x^{2n},$$

from which

$$x^3 S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{(2n)!} x^{2n+3}.$$

Integration gives

$$\int x^3 S(x)\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{(2n)!(2n+4)} x^{2n+4} + C = \frac{x^4}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + C = \frac{x^4}{2} \cos x + C.$$

Differentiation now gives

$$x^3 S(x) = 2x^3 \cos x - \frac{x^4}{2} \sin x \quad \Longrightarrow \quad S(x) = 2\cos x - \frac{x}{2} \sin x,$$

valid for $-\infty < x < \infty$.

10 **2.** The Maclaurin series for $\ln(1+2x)$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^n}{n} x^n.$$

Find the maximum error when $\ln(1+2x)$ is approximated by the first four terms of this series on the interval $0 \leq x \leq 0.1$.

The series is alternating for $0 \leq x \leq 0.1$, and for these values of x, the sequence $\left\{\frac{2^n x^n}{n}\right\}$ is decreasing and has limit zero. The series therefore converges by the alternating series test, and the maximum error in truncating the series after the first four terms is the absolute value of the fifth term,

$$\left|\frac{(-1)^6 2^5 x^5}{5}\right| \leq \frac{2^5(0.1)^5}{5}.$$

8 **3.** Find a one-parameter family of solutions of the differential equation

$$y\frac{dy}{dx} = (y+1)\sin 3x + 2\frac{dy}{dx}.$$

Does your family have any singular solutions? Explain.

$$(y-2)\frac{dy}{dx} = (y+1)\sin 3x$$
$$\frac{y-2}{y+1}dy = \sin 3x\, dx, \qquad y \neq -1$$

A one-parameter family of solutions is defined implicitly by

$$\int \frac{y-2}{y+1}dy = \int \sin 3x\, dx$$
$$\int \left(1 - \frac{3}{y+1}\right) dy = -\frac{1}{3}\cos 3x + C$$
$$y - 3\ln|y+1| = -\frac{1}{3}\cos 3x + C$$

$y = -1$ is a solution of the given differeential equaation. Because it cannot be obtained from the one-parameter family by specifying a value for C, it is a singular solution of the family.

12 **4.** A tank originally contains 1000 litres of water in which 5 kilograms of salt has been dissolved. A mixture containing 2 kilograms of salt for each 100 litres of solution is added to the tank at 10 millilitres per second. At the same time, the well-stirred mixture in the tank is removed at the rate of 5 millilitres per second. Find the amount of salt in the tank as a function of time.

Let $S(t)$ represent the number of grams of salt in the tank as a function of time t in seconds. Then

$$\begin{aligned}\frac{dS}{dt} &= \text{(rate salt enters)} - \text{(rate salt leaves)}\\ &= \left(\frac{2000}{10^5}\right)(10) - \frac{S}{10^6+5t}(5)\\ &= \frac{1}{5} - \frac{5S}{10^6+5t},\end{aligned}$$

subject to $S(0) = 5000$. An integrating factor is

$$e^{\int 5/(10^6+5t)dt} = e^{\ln(10^6+5t)} = 10^6 + 5t.$$

Multlliplying terms of the differential equation by this gives

$$\begin{aligned}(10^6+5t)\frac{dS}{dt} + 5S &= \frac{1}{5}(10^6+5t)\\ \frac{d}{dt}[S(10^6+5t)] &= \frac{1}{5}(10^6+5t)\\ S(10^6+5t) &= \frac{1}{50}(10^6+5t)^2 + C\\ S(t) &= \frac{1}{50}(10^6+5t) + \frac{C}{10^6+5t}\end{aligned}$$

The initial condition requires

$$5000 = \frac{10^6}{50} + \frac{C}{10^6} \quad \Longrightarrow \quad C = -15 \times 10^9.$$

Thus,

$$S(t) = \frac{1}{50}(10^6+5t) - \frac{15\times 10^9}{10^6+5t}\ \text{g}.$$