MATH2132 Test 2 **March 6, 2018** **60 minutes**

Student Name - **Student Number -**

Values

12 **1.** Find the sum of the power series

$$\sum_{n=0}^{\infty} \frac{1}{3^n(n+1)} x^n.$$

What is the interval of validity of your sum?

The radius of convergence of the series is $R = \lim_{n\to\infty} \left| \dfrac{\dfrac{1}{3^n(n+1)}}{\dfrac{1}{3^{n+1}(n+2)}} \right| = 3.$

If we set $S(x) = \sum_{n=0}^{\infty} \frac{1}{3^n(n+1)} x^n$, then

$$x\,S(x) = \sum_{n=0}^{\infty} \frac{1}{3^n(n+1)} x^{n+1}.$$

Differentiation with respect to x gives

$$\frac{d}{dx}[x\,S(x)] = \sum_{n=0}^{\infty} \frac{1}{3^n} x^n = \frac{1}{1-x/3} = \frac{3}{3-x},$$

provided $|x/3| < 1 \implies -3 < x < 3$. Integration now gives

$$x\,S(x) = \int \frac{3}{3-x} dx = -3\ln(3-x) + C.$$

If we set $x = 0$, then $0 = -3\ln 3 + C$. Thus,

$$x\,S(x) = -3\ln(3-x) + 3\ln 3,$$

from which

$$S(x) = \frac{3}{x}\ln 3 - \frac{3}{x}\ln(3-x), \quad \text{provided } x \neq 0.$$

Since $S(0) = 1$,

$$S(x) = \begin{cases} \dfrac{3}{x}\ln 3 - \dfrac{3}{x}\ln(3-x), & -3 < x < 3,\ x \neq 0 \\ 1, & x = 0. \end{cases}$$

10 **2.** Find the maximum possible error in using the first three nonzero terms in the Taylor series for $f(x) = \sqrt{1+3x}$ about $x = 1$ to approximate the function on the interval $1 \le x \le 3/2$. Justify all steps in your solution.

Method 1 - Alternating series

The binomial expansion gives

$$f(x) = [3(x-1)+4]^{1/2} = 2\left[1 + \frac{3(x-1)}{4}\right]^{1/2}$$
$$= 2\left[1 + \frac{1}{2}\left(\frac{3}{4}\right)(x-1) + \frac{(1/2)(-1/2)}{2!}\left(\frac{3}{4}\right)^2 (x-1)^2 + \cdots\right].$$

For values $1 \le x \le 3/2$, the series is alternating after the first term. In addition, absolute values of the terms are decreasing and have limit zero. The series therefore converges by the alternating series test, and the maximum error in using the above three terms to approximate the function is the absolute value of the next term

$$2\left|\frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{3}{4}\right)^3 (x-1)^3\right| \le \frac{2(3^4)}{3!\,2^9}\left(\frac{1}{2}\right)^3 = \frac{3^3}{2^{12}}.$$

Method 2 - Taylor remainders

The error in using the above three terms to approxiate the function is

$$|R_2(1,x)| = \left|\frac{f'''(z)(x-1)^3}{3!}\right| = \left|\frac{(1/2)(-1/2)(-3/2)3^3}{(1+3z)^{5/2}}\frac{(x-1)^3}{3!}\right|.$$

Since $1 < z < x \le 3/2$,

$$|R_2(1,x)| \le \frac{3^4}{2^3(4)^{5/2}}\frac{|x-1|^3}{3!} \le \frac{3^4}{2^8(3!)}\left(\frac{1}{2}\right)^3 = \frac{3^3}{2^{12}}.$$

12 **3.** Solve the initial-value problem

$$\frac{dy}{dx} = (y-2)(y-3), \qquad y(0) = 1.$$

The equation is separable,

$$dx = \frac{dy}{(y-2)(y-3)},$$

and a one-parameter family of solutions is defined implicitly by

$$x + C = \int \left(\frac{-1}{y-2} + \frac{1}{y-3} \right) dy = -\ln|y-2| + \ln|y-3|.$$

Thus,

$$\ln\left|\frac{y-3}{y-2}\right| = x + C,$$

and exponentiation gives

$$\left|\frac{y-3}{y-2}\right| = e^{x+C} \quad \Longrightarrow \quad \frac{y-3}{y-2} = De^x, \quad \text{where } D = \pm e^C.$$

Thus,

$$\begin{aligned} y - 3 &= (y-2)De^x \\ y(1 - De^x) &= 3 - 2De^x \\ y(x) &= \frac{3 - 2De^x}{1 - De^x}. \end{aligned}$$

Since $y(0) = 1$,

$$1 = \frac{3 - 2D}{1 - D} \quad \Longrightarrow \quad 1 - D = 3 - 2D \quad \Longrightarrow \quad D = 2.$$

Finally then

$$y(x) = \frac{3 - 4e^x}{1 - 2e^x}.$$

6 **4.** When a deep-sea diver inhales air, his body absorbs extra amounts of nitrogen. Suppose the diver enters the water at time $t = 0$, drops very quickly to depth d, and remains at this depth for a long time. The amount of nitrogen in his body increases as he remains at this depth until a maximum amount $\overline{N}$ is reached. If the amount of nirtogen in his body increases at a rate proportional to the difference between $\overline{N}$ and the present amount of nitrogen in his body, set up, but do **NOT** solve, an initial-value problem for the amount of nitrogen in his body as he remains at depth d.

Let $N(t)$ be the amount of nitrogen in the diver's body where t is time, and take $t = 0$ when he arrives at depth d. Then

$$\frac{dN}{dt} = k(\overline{N} - N).$$

If N_0 is the amount of nitrogen in his body when he enters the water, $N(t)$ is also subject to the initial condition $N(0) = N_0$.