MATH2132 Test 3 **November 21, 2013** **70 minutes**

Student Name - **Student Number -**

Values

10 **1.** (a) Find an explicitly defined, one-parameter family of solutions for the differential equation

$$x\frac{dy}{dx} = (1-y)(x+1).$$

Simplify your solution as much as possible.

(b) Show that your family has a singular solution. Can this solution be added to the family so that it is no longer singular?

(a) The equation can be separated,

$$\frac{dy}{1-y} = \frac{x+1}{x}dx = \left(1+\frac{1}{x}\right)dx, \qquad y \neq 1.$$

A one-parameter family of solutions is defined implicitly by

$$\begin{aligned}
\int \frac{1}{1-y}dy &= \int \left(1+\frac{1}{x}\right)dx \\
-\ln|1-y| &= x + \ln|x| + C \\
\ln|1-y| &= -x - \ln|x| - C \\
|1-y| &= e^{-x-\ln|x|-C} \\
1-y &= \pm e^C x e^{-x} \\
y &= 1 - Dxe^{-x}, \quad D = \pm e^C
\end{aligned}$$

(b) Since $y = 1$ is a solution of the differential equation, and it is not in the family ($D = \pm e^C \neq 0$), it is a singular solution of the family. If, however, we allow D to be zero, then this solution is contained in the family, and it is no longer singular.

10 **2.** Find a general solution of the differential equation

$$x\frac{dy}{dx} = 2y + x^4 e^{-x}.$$

This equation is linear, first-order,

$$\frac{dy}{dx} - \frac{2}{x}y = x^3 e^{-x}.$$

An integrating factor is

$$e^{\int (-2/x)dx} = e^{-2\ln|x|} = \frac{1}{x^2}.$$

We multiply each term of the equation by this factor,

$$\begin{aligned} \frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y &= xe^{-x} \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= xe^{-x} \\ \frac{y}{x^2} &= \int xe^{-x}dx + C = -xe^{-x} - e^{-x} + C. \end{aligned}$$

A general solution is

$$y(x) = -x^2(x+1)e^{-x} + Cx^2.$$

13 **3.** Find a general solution for the differential equation

$$3y''' + 2y'' + 2y' - y = 2e^x - x.$$

The auxiliary equation is

$$0 = 3m^3 + 2m^2 + 2m - 1 = (3m-1)(m^2 + m + 1).$$

Roots are

$$m = \frac{1}{3}, \qquad m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

A general solution of the associated homogeneous equation is

$$y_h(x) = C_1 e^{x/3} + e^{-x/2}\left(C_2 \cos\frac{\sqrt{3}x}{2} + C_3 \sin\frac{\sqrt{3}x}{2}\right).$$

A particular solution is of the form $y_p(x) = Ae^x + Bx + C$. Substitution into the differential equation gives

$$3(Ae^x) + 2(Ae^x) + 2(Ae^x + B) - (Ae^x + Bx + C) = 2e^x - x.$$

When we equate coefficients:

$$\begin{aligned} e^x: &\quad 3A + 2A + 2A - A = 2 \quad \Longrightarrow \quad A = \frac{1}{3} \\ x: &\quad -B = -1 \quad \Longrightarrow \quad B = 1 \\ 1: &\quad 2B - C = 0 \quad \Longrightarrow \quad C = 2 \end{aligned}$$

Thus, $y_p(x) = \frac{1}{3}e^x + x + 2$, and

$$y(x) = C_1 e^{x/3} + e^{-x/2}\left(C_2 \cos\frac{\sqrt{3}x}{2} + C_3 \sin\frac{\sqrt{3}x}{2}\right) + \frac{1}{3}e^x + x + 2.$$

7 **4.** (a) You are given that a general solution of the homogeneous equation associated with the linear, constant coefficient differential equation

$$\phi(D)y = x^2 + e^x \cos 2x$$

is

$$y_h(x) = C_1 + C_2 x + C_3 \cos 2x + C_4 \sin 2x.$$

What is the differential equation?

(b) What is the form for a particular solution of the differential equation as predicted by the method of undetermined coefficients? Do **NOT** evaluate the coefficients.

(a) The roots of the auxiliary equation leading to $y_h(x)$ are 0,0,$\pm 2i$. Hence, $\phi(m) = m^2(m^2+4) = m^4 + 4m^2$. The differential equation is

$$y'''' + 4y'' = x^2 + e^x \cos 2x.$$

(b) $y_p(x) = Ax^4 + Bx^3 + Cx^2 + De^x \cos 2x + Ee^x \sin 2x.$