MATH2132 Test 3 **March 25, 2014** **70 minutes**

Student Name - **Student Number -**

Values

12 **1.** (a) Solve the initial value problem

$$x^2y'' + 2(y')^2 = 0, \qquad y(1) = 2, \qquad y'(1) = 1.$$

If we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$, then

$$x^2\frac{dv}{dx} + 2v^2 = 0 \qquad \Longrightarrow \qquad \frac{dv}{v^2} = -2\frac{dx}{x^2}, \quad (v \neq 0).$$

A one-parameter family of solutions of this separated equation is defined implicitly by

$$-\frac{1}{v} = \frac{2}{x} + C \qquad \Longrightarrow \qquad v = \frac{-x}{2+Cx}.$$

The condition $y'(1) = 1$ implies that

$$1 = \frac{-1}{2+C} \qquad \Longrightarrow \qquad 2 + C = -1 \qquad \Longrightarrow \qquad C = -3.$$

Thus,

$$\frac{dy}{dx} = v = \frac{-x}{2-3x} = \frac{x}{3x-2}.$$

Integration gives

$$y = \int \frac{x}{3x-2}dx = \int \left(\frac{1}{3} + \frac{2/3}{3x-2}\right) dx = \frac{x}{3} + \frac{2}{9}\ln|3x-2| + D.$$

Since $y(1) = 2$,

$$2 = \frac{1}{3} + \frac{2}{9}\ln 1 + D \qquad \Longrightarrow \qquad D = \frac{5}{3}.$$

The solution is therefore

$$y(x) = \frac{x}{3} + \frac{2}{9}\ln|3x-2| + \frac{5}{3}.$$

12 **2.** Find a general solution of the differential equation

$$4y''' + 5y'' + 9y' + 2y = e^{x/4}.$$

The auxiliary equation is

$$0 = 4m^3 + 5m^2 + 9m + 2 = (4m+1)(m^2+m+2),$$

with roots $m = -1/4$ and $m = \dfrac{-1 \pm \sqrt{1-8}}{2} = \dfrac{-1 \pm \sqrt{7}i}{2}$. A general solution of the associated homogeneous equation is

$$y_h(x) = C_1 e^{-x/4} + e^{-x/2}\left(C_2 \cos\frac{\sqrt{7}x}{2} + C_3 \sin\frac{\sqrt{7}x}{2}\right).$$

A particular solution is of the form $y_p(x) = Ae^{x/4}$. Substitution into the differential equation gives

$$4A\left(\frac{1}{64}e^{x/4}\right) + 5A\left(\frac{1}{16}e^{x/4}\right) + 9A\left(\frac{1}{4}e^{x/4}\right) + 2Ae^{x/4} = e^{x/4}.$$

This implies that

$$\frac{A}{16} + \frac{5A}{16} + \frac{9A}{4} + 2A = 1 \qquad \Longrightarrow \qquad A = \frac{8}{37}.$$

Thus, $y_p(x) = \dfrac{8}{37}e^{x/4}$, and

$$y(x) = C_1 e^{-x/4} + e^{-x/2}\left(C_2 \cos\frac{\sqrt{7}x}{2} + C_3 \sin\frac{\sqrt{7}x}{2}\right) + \frac{8}{37}e^{x/4}.$$

6 **3.** If the roots of the auxiliary equation $\phi(m) = 0$ associated with the differential equation

$$\phi(D)y = x^2 - 4e^{2x} + 2\cos 4x$$

are $m = 0$, $3 \pm 4i$, ± 2, ± 2, $\pm\sqrt{3}i$, state the form of a particular solution $y_p(x)$ as predicted by the method of undetermined coefficients. Do **NOT** evaluate the coefficients.

$$\begin{aligned} y_h(x) = C_1 + e^{3x}(C_2\cos 4x + C_3 \sin 4x) + (C_4 + C_5 x)e^{2x} + (C_6 + C_7 x)e^{-2x} \\ + C_8 \cos\sqrt{3}x + C_9 \sin\sqrt{3}x. \end{aligned}$$

$$y_p(x) = Ax^3 + Bx^2 + Cx + Dx^2e^{2x} + E\cos 4x + F\sin 4x.$$

11 **4.** (a) A 1 kilogram mass is suspended from a spring with constant 20 newtons per metre. It is set into motion by pulling it 10 centimetres below its equilibrium position and releasing it. During its subsequent motion, it is subjected to a damping force that is equal to 12 times its velocity.

(a) Find the position of the mass as a function of time t.

(b) Determine all times at which the mass passes through its equilibrium position, or prove that it never passes through the equilibrium position.

(a) The initial-value problem for displacement $x(t)$ from equilibrium is

$$1\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 20x = 0, \quad x(0) = -\frac{1}{10}, \quad x'(0) = 0.$$

The auxiliary equation is

$$0 = m^2 + 12m + 20 = (m+2)(m+10) \qquad \Longrightarrow \qquad m = -2, -10.$$

A general solution of the differential equation is therefore

$$x(t) = C_1e^{-2t} + C_2e^{-10t}.$$

The initial conditions require

$$-\frac{1}{10} = C_1 + C_2, \quad 0 = -2C_1 - 10C_2 \qquad \Longrightarrow \qquad C_1 = -\frac{1}{8}, \quad C_2 = \frac{1}{40}.$$

Thus,

$$x(t) = -\frac{1}{8}e^{-2t} + \frac{1}{40}e^{-10t} \text{ m}.$$

(b) The mass passes through equilibrium if, and when,

$$0 = -\frac{1}{8}e^{-2t} + \frac{1}{40}e^{-10t} \quad \Longrightarrow \quad e^{8t} = \frac{1}{5} \quad \Longrightarrow \quad t = \frac{1}{8}\ln(1/5) = -\frac{1}{8}\ln 5.$$

Since t must be positive, the mass never passes through equilibrium.