MATH 2132 Tutorial 10 - Solutions

1. A 500-gram mass is placed on a table and attached to a spring with constant 20 newtons per metre. The other end of the spring is attached to a wall. The mass is pushed 5 centimetres so as to compress the spring, and then released. The coefficient of kinetic friction between the mass and table is $\mu = 0.2$. Find where the mass stops moving for the first time. Does it move from this position?

Solution: Let $x = x(t)$ be the (horizontal) position of the mass at time t, where x is considered negative when the spring is compressed, and x is considered positive when the spring is stretched. Thus, at time $t = 0$, $x = x(0) = -0.05\,\text{m}$. Also, $x'(t) = 0$, since no initial velocity is given.

The function $x(t)$ satisfies the differential equation

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t)\,,$$

where $M = 0.5\,\text{kg}$, $\beta = 0$, $k = 20\,\text{N}$, and $F(t)$ represents the friction force. The magnitude of the friction force is $\mu(\text{gravity force}) = \mu Mg = (0.2)(0.5)g = 0.1g$ newtons. However, if the spring is compressed, then the spring force acts in the positive direction, and so the opposing friction force is $F(t) = -0.1g$ newtons. Therefore $x(t)$ satisfies the following:

$$0.5\frac{d^2x}{dt^2} + 20x = -0.1g\,, \quad x(0) = 0.05\,, \quad x'(0) = 0\,. \tag{1}$$

Note that this initial-value problem is valid for as long as the mass is moving in the positive direction. This means that if the mass stops moving in the positive direction for the first time at $t = t_0$, then any solution $x(t)$ of the initial-value problem is only valid over the interval $0 \le t \le t_0$. Should the mass ever start moving in the negative direction, the friction force would change to $F(t) = +0.1g$ newtons.

The initial value problem (1) is equivalent to

$$\frac{d^2x}{dt^2} + 40x = -0.2g\,, \quad x(0) = 0.05\,, \quad x'(0) = 0\,.$$

The auxiliary equation is $m^2 + 40 = 0$ with solutions $m = \pm 2\sqrt{10}i$. Consequently, a general solution of the corresponding homogeneous equation is

$$x_h(t) = C_1\cos(2\sqrt{10}t) + C_2\sin(2\sqrt{10}t)\,.$$

A particular solution would have to be a constant function $x(t) = A$. Substitution of this function into the differential equation gives $40A = -0.2g$. Hence $A = -g/200$, and so $x_p(t) = -g/200$. Thus a general solution is

$$x(t) = x_p(t) + x_h(t) = -\frac{g}{200} + C_1 \cos(2\sqrt{10}t) + C_2 \sin(2\sqrt{10}t).$$

The condition $x(0) = -0.05$ implies

$$-0.05 = -\frac{g}{200} + C_1.$$

Hence $C_1 = g/200 - 0.05 = (g - 10)/200$. Since $x'(t) = -2\sqrt{10}C_1 \sin(2\sqrt{10}t) + 2\sqrt{10}C_2 \cos(2\sqrt{10}t)$, the condition $x'(0) = 0$ implies

$$0 = 2\sqrt{10}C_2.$$

Therefore $C_2 = 0$, and so the final solution is

$$x(t) = -\frac{g}{200} + \frac{g-10}{200} \cos(2\sqrt{10}t).$$

Notice that the coefficient $(g - 10)/200$ is a negative number, since $g \approx 9.81 < 10$. Thus, starting from $t = 0$, our solution $x(t)$ is an increasing function, until $\cos 2\sqrt{10}t$ reaches its minimum value -1. This occurs when $2\sqrt{5}t = \pi$, or $t = \pi/2\sqrt{10}$. At this moment, $x = x(t) = -g/200 - (g - 10)/200 = (-2g + 10)/200 = (5 - g)/100$. Since the value $x = (5 - g)/100$ is negative, the farthest position the mass will reach takes place while the spring is still compressed. The position $x = (5 - g)/100$ represents the equilibrium, where the spring force is equal to the opposing friction force.

Therefore the mass will stop moving when it reaches the position $x = (5 - g)/100$.

2. (a) A 2-kilogram mass is suspended from a spring with constant 1000 newtons per metre. A force $2 \sin \omega t$ newtons initiates motion at time $t = 0$, and continues to act on the mass. Find the position of the mass as a function of time when resonance does not occur.

(b) What value of ω causes resonance?

<u>Solution:</u>

(a) By setting $y = 0$ at the equilibrium, we may ignore the gravity force. The position

$y = y(t)$ of the mass at time t satisfies the initial-value problem:

$$2\frac{d^2y}{dt^2} + 1000y = 2\sin\omega t\,, \quad y(0) = 0\,, \quad y'(0) = 0\,,$$

or equivalently,

$$\frac{d^2y}{dt^2} + 500y = \sin\omega t\,, \quad y(0) = 0\,, \quad y'(0) = 0\,. \tag{2}$$

The auxiliary equation is $m^2+500 = 0$ with solutions $m = \pm 10\sqrt{5}i$. Consequently, a general solution of the corresponding homogeneous equation is

$$y_h(t) = C_1\cos(10\sqrt{5}t) + C_2\sin(10\sqrt{5}t)\,.$$

Note that the function $F(t) = \sin\omega t$ can be obtained from $y_h(t)$ if $\omega = 10\sqrt{5}$. On the other hand, if $\omega \neq 10\sqrt{5}$, then $F(t) = \sin\omega t$ cannot be obtained from $y_h(t)$.

Case 1 $\quad \omega \neq 10\sqrt{5}$

In this case, we can take a particular solution of the form

$$y_p(t) = A\cos\omega t + B\sin\omega t\,.$$

Substitution of this function into the differential equation gives

$$(-\omega^2 A\cos\omega t - \omega^2 B\sin\omega t) + 500(A\cos\omega t + B\sin\omega t) = \sin\omega t\,.$$

Hence

$$(500-\omega^2)A\cos\omega t + (500-\omega^2)B\sin\omega t = \sin\omega t\,,$$

and so

$$(500-\omega^2)A = 0 \qquad \text{and} \qquad (500-\omega^2)B = 1\,.$$

Therefore

$$A = 0 \qquad \text{and} \qquad B = \frac{1}{500-\omega^2}\,.$$

Thus a general solution of the differential equation in (2) is

$$y(t) = y_p(t) + y_h(t) = \frac{1}{500-\omega^2}\sin\omega t + C_1\cos(10\sqrt{5}t) + C_2\sin(10\sqrt{5}t)\,.$$

The condition $y(0)=0$ implies $C_1=0$. Hence

$$y'(t)=\frac{\omega}{500-\omega^2}\cos\omega t+10\sqrt{5}C_2\cos(10\sqrt{5}t)\,.$$

Now the condition $y'(0)=0$ implies

$$0=\frac{\omega}{500-\omega^2}+10\sqrt{5}C_2\,.$$

Hence

$$C_2=\frac{-\omega}{10\sqrt{5}(500-\omega^2)}\,.$$

Therefore the final solution is

$$y(t)=\frac{1}{500-\omega^2}\sin\omega t-\frac{\omega}{10\sqrt{5}(500-\omega^2)}\sin(10\sqrt{5}t)\,. \tag{3}$$

Since $\omega\neq 10\sqrt{5}$, the two sine functions have different phases. Note that each of the two terms in (3) can have a very large amplitude when the expression $500-\omega^2$ in the denominator is very small. This occurs when ω is close to the value $10\sqrt{5}$. However, both amplitudes are constant, as t increases, no matter how large the amplitudes may be.

<u>Case 2</u> $\quad\omega=10\sqrt{5}$

Since the roots of the auxiliary equation are of multiplicity 1, we can take a particular solution of the form

$$y_p(t)=At\cos(10\sqrt{5}t)+Bt\sin(10\sqrt{5}t)\,.$$

Hence

$$\frac{d^2y_p}{dt^2}=-20\sqrt{5}A\sin(10\sqrt{5}t)+20\sqrt{5}B\cos(10\sqrt{5}t)-500At\cos(10\sqrt{5}t)-500Bt\sin(10\sqrt{5}t)\,.$$

Substitution into (2) gives

$$-20\sqrt{5}A\sin(10\sqrt{5}t)+20\sqrt{5}B\cos(10\sqrt{5}t)=\sin(10\sqrt{5}t)\,.$$

Therefore

$$-20\sqrt{5}A=1 \qquad\text{and}\qquad 20\sqrt{5}B=0\,.$$

Hence

$$A = \frac{-1}{20\sqrt{5}} \qquad \text{and} \qquad B = 0\,.$$

Thus a general solution of the differential equation in (2) is

$$y(t) = y_p(t) + y_h(t) = \frac{-t}{20\sqrt{5}}\cos\left(10\sqrt{5}t\right) + C_1\cos\left(10\sqrt{5}t\right) + C_2\sin\left(10\sqrt{5}t\right).$$

Now the condition $y(0) = 0$ implies $C_1 = 0$. Hence

$$y'(t) = \frac{-1}{20\sqrt{5}}\cos\left(10\sqrt{5}t\right) + \frac{t}{2}\sin\left(10\sqrt{5}t\right) + 10\sqrt{5}C_2\cos\left(10\sqrt{5}t\right).$$

The condition $y'(0) = 0$ implies

$$0 = \frac{-1}{20\sqrt{5}} + 10\sqrt{5}C_2\,.$$

Therefore

$$C_2 = \frac{1}{1000}\,.$$

The final solution, for the case $\omega = 10\sqrt{5}$ is

$$y(t) = \frac{-t}{20\sqrt{5}}\cos\left(10\sqrt{5}t\right) + \frac{1}{1000}\sin\left(10\sqrt{5}t\right).$$

Notice that the amplitude of the second term, involving a sine function, is constant and is equal to 0.001. However, the amplitude of the first term, involving a cosine function, is not constant. In fact, its amplitudes increase linearly, without a bound, as t increases to infinity.

In conclusion, the resonance occurs when $\omega = 10\sqrt{5}$. The final solution for the case when the resonance does not occur is given by (3).

(b) By the analysis in part (a), the value $\omega = 10\sqrt{5}$ maximizes the amplitudes of the oscillations, thus causing resonance.

3. Repeat part (a) of problem 2 if a damping force proportional to velocity with $\beta = 10$ acts on the mass.

Solution: The relevant initial-value problem is:

$$2\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 1000y = 2\sin\omega t\,, \quad y(0) = 0\,, \quad y'(0) = 0\,,$$

or equivalently,

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 500y = \sin\omega t\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$

The auxiliary equation is $m^2 + 5m + 500 = 0$ with solutions $m = (-5/2) \pm (5\sqrt{79}/2)i$. Consequently, a general solution of the corresponding homogeneous equation is

$$y_h(t) = C_1 e^{-5t/2} \cos\frac{5\sqrt{79}}{2}t + C_2 e^{-5t/2} \sin\frac{5\sqrt{79}}{2}t\,.$$

Note that the function $F(t) = \sin\omega t$ cannot be obtained from $y_h(t)$. Therefore we can take a particular solution of the form

$$y_p(t) = A\cos\omega t + B\sin\omega t\,.$$

Substitution of this function into the differential equation gives

$$(-\omega^2 A\cos\omega t - \omega^2 B\sin\omega t) + 5(-\omega A\sin\omega t + \omega B\cos\omega t) + 500(A\cos\omega t + B\sin\omega t) = \sin\omega t\,,$$

or equivalently,

$$(-\omega^2 A + 5\omega B + 500A)\cos\omega t + (-\omega^2 B - 5\omega A + 500B)\sin\omega t = \sin\omega t\,.$$

Hence

$$-\omega^2 A + 5\omega B + 500A = 0 \qquad \text{and} \qquad -\omega^2 B - 5\omega A + 500B = 1\,,$$

or equivalently,

$$\begin{aligned} (500 - \omega^2)A + 5\omega B &= 0 \\ -5\omega A + (500 - \omega^2)B &= 1\,. \end{aligned}$$

This linear system can be solved for A and B, say by Cramer's rule. The unique solution is

$$A = \frac{-5\omega}{(500 - \omega^2)^2 + 25\omega^2} \qquad \text{and} \qquad B = \frac{500 - \omega^2}{(500 - \omega^2)^2 + 25\omega^2}\,.$$

Thus a general solution of the differential equation in (3) is

$$\begin{aligned}
y(t) &= y_p(t) + y_h(t) \\
&= \frac{-5\omega}{(500-\omega^2)^2+25\omega^2}\cos\omega t + \frac{500-\omega^2}{(500-\omega^2)^2+25\omega^2}\sin\omega t \\
&\quad + C_1 e^{-5t/2}\cos\frac{5\sqrt{79}}{2}t + C_2 e^{-5t/2}\sin\frac{5\sqrt{79}}{2}t\,.
\end{aligned}$$

Now the condition $y(0) = 0$ implies

$$0 = \frac{-5\omega}{(500-\omega^2)^2+25\omega^2} + C_1\,.$$

Therefore

$$C_1 = \frac{5\omega}{(500-\omega^2)^2+25\omega^2}\,.$$

Since

$$\begin{aligned}
y'(t) &= \frac{5\omega^2}{(500-\omega^2)^2+25\omega^2}\sin\omega t + \frac{\omega(500-\omega^2)}{(500-\omega^2)^2+25\omega^2}\cos\omega t \\
&\quad - \frac{5C_1}{2}e^{-5t/2}\cos\frac{5\sqrt{79}}{2}t - \frac{5\sqrt{79}C_1}{2}e^{-5t/2}\sin\frac{5\sqrt{79}}{2}t \\
&\quad - \frac{5C_2}{2}e^{-5t/2}\sin\frac{5\sqrt{79}}{2}t + \frac{5\sqrt{79}C_2}{2}e^{-5t/2}\cos\frac{5\sqrt{79}}{2}t\,,
\end{aligned}$$

the condition $y'(0) = 0$ implies

$$0 = \frac{\omega(500-\omega^2)}{(500-\omega^2)^2+25\omega^2} - \frac{5C_1}{2} + \frac{5\sqrt{79}C_2}{2}\,.$$

Hence

$$\begin{aligned}
C_2 &= \frac{-2\omega(500-\omega^2)}{5\sqrt{79}[(500-\omega^2)^2+25\omega^2]} + \frac{C_1}{\sqrt{79}} \\
&= \frac{2\omega^3-1000\omega)}{5\sqrt{79}[(500-\omega^2)^2+25\omega^2]} + \frac{5\omega}{\sqrt{79}[(500-\omega^2)^2+25\omega^2]} \\
&= \frac{\omega(2\omega^2-975)}{5\sqrt{79}[(500-\omega^2)^2+25\omega^2]}\,.
\end{aligned}$$

Therefore the final solution is

$$
\begin{aligned}
y(t) = {} & \frac{-5\omega}{(500-\omega^2)^2+25\omega^2}\cos\omega t + \frac{500-\omega^2}{(500-\omega^2)^2+25\omega^2}\sin\omega t \\
& + e^{-5t/2}\left[\frac{5\omega}{(500-\omega^2)^2+25\omega^2}\cos\frac{5\sqrt{79}}{2}t + \frac{\omega(2\omega^2-975)}{5\sqrt{79}[(500-\omega^2)^2+25\omega^2]}\sin\frac{5\sqrt{79}}{2}t\right].
\end{aligned}
$$