CHAPTER 6 LAPLACE TRANSFORMS

The Laplace transform is one of many *integral transforms* in applied mathematics. Through an improper integral, the Laplace transform creates an association between a class of functions denoted by $f(t)$ and a class of functions denoted by $F(s)$. The advantage of this association as far as our discussions are concerned is that solving a differential equation for $f(t)$ is replaced by solving an algebraic equation for $F(s)$. The fact that the Laplace transform is a linear operator (in the sense of equation 4.12) makes it particularly useful for solving the linear differential equations encountered in Chapters 4 and 5. Furthermore, you will recall that in Chapter 4 we assumed continuity of nonhomogeneous terms in linear differential equations. This was a matter of convenience rather than necessity. In Exercises 32 and 33 of Section 4.5, we hinted at the awkwardness of incorporating discontinuities into the techniques of Chapter 4. But discontinuous nonhomogeneities occur frequently in applications. For instance, in Example 3.10 of Section 3.4, the nonhomogeneity 1/10 in the linear differential equation

$$\frac{dS}{dt} + \frac{5S}{10^6} = \frac{1}{10}$$

is a result of brine with concentration 2 kilograms per 100 litres being added to the tank. But suppose after 10 minutes the concentration is doubled to 4 kilograms per 100 litres, and after another 10 minutes it is increased to 5 kilograms per 100 litres. The nonhomogenity would now be

$$f(t) = \begin{cases} 1/10, & 0 < t < 600 \\ 1/5, & 600 < t < 1200 \\ 1/4, & t > 1200. \end{cases}$$

It is possible to solve this problem with techniques from Chapter 4, but it isn't very convenient to do so. We shall show how easily such situations are handled by Laplace transforms. An even more awkward situation for Chapter 4 would be an LCR-circuit where the applied voltage is continually turned on for a second, turned off for another second, turned on again, turned off again, ad infinitum. Once again this presents no problem for Laplace transforms. Even more devastating for Chapter 4 is imparting an instantaneous force to a vibrating mass-spring system say by striking the mass with a hammer. Chapter 4 just cannot handle this situation, but Laplace transforms can. This is perhaps the biggest advantage of Laplace transforms over the methods of Chapter 4. Discontinuous rates in mixing problems, discontinuous forcing functions in vibrating mass-spring systems, and discontinuous driving voltages in LCR-circuits are easily handled by Laplace transforms. Furthermore, nonhomogeneities that represent "point" concepts in space or time cannot be handled with the techniques of Chapter 4, but they present no problem for Laplace transforms. Such occurrences include bulk additions of ingredients in mixing problems, instantaneously applied forces in vibrating problems, instantaneously applied voltages in electric circuits, and point loads on beams.

We make one last point before defining the Laplace transform of a function. Currents in an electrical network satisfy a system of interrelated differential equations, a topic that we take up in Chapter 7. Laplace transforms of these equations reduce them to a system of algebraic equations. Examination of the functions in these equations can determine whether the network is stable or whether it would lead to unacceptably high currents, and this can be done without actually solving for the currents.

§6.1 The Laplace Transform and its Inverse

The Laplace transform is one of many useful integral transforms in mathematics. They all map a function of one variable to a function of a different variable. In general, an integral transform maps a function $f(t)$ of t to a function $F(s)$ of s according to the definition

$$F(s) = \int_a^b K(t,s)f(t)\,dt. \tag{6.1}$$

The function $K(t,s)$ is called the **kernel** of the transformation (and it is given). The function $F(s)$ is called the transform of $f(t)$, and to find $F(s)$, we must multiply $f(t)$ by $K(t,s)$ and integrate the product from a to b. According to the following definition, the kernel of the Laplace transform is $K(t,s) = e^{-st}$, and limits are $a = 0$ and $b = \infty$.

Definition 6.1 When f is a function of t, its **Laplace transform** denoted by $F = \mathcal{L}\{f\}$ is a function with values defined by

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st}f(t)\,dt, \tag{6.2}$$

provided the improper integral converges.

The Laplace transform of a function $f(t)$ never exists for all values of s; there is always a restriction on the values that can be used. In other words, the real problem is to find values of s for which improper integral 6.2 converges, and for these values, and only these values, $F(s)$ is the Laplace transform of $f(t)$.

It is customary to choose t as the independent variable of functions that are to be transformed because the transform is so often associated with problems in which t represents time. Do not get the impression, however, that Laplace transforms are only associated with time. In Section 6.6, we take Laplace transform with respect to x in order to calculate deflections of beams under various loads. It is customary to use a lower case letter f to represent a function that is to be transformed, and its capital counterpart F to represent the transform of f.

When t is time, with units of seconds, then s must have units of one divided by seconds. If this were not the case, then $-st$ in e^{-st} would not be dimensionless, and what then would be the units of the exponential. Hence, the Laplace transform is a mapping from the time domain to the frequency domain.

The improper integral in Definition 6.1 is defined as

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \lim_{T\to\infty}\int_0^T e^{-st}f(t)\,dt,$$

provided the limit exists (and also provided $f(t)$ has acceptable discontinuities, further discussion coming). If $g(s,t)$ is an antiderivative of $e^{-st}f(t)$ with respect to t, we could write

$$F(s) = \lim_{T\to\infty} \{g(s,t)\}_0^T = \lim_{T\to\infty} [g(s,T) - g(s,0)] = \lim_{T\to\infty} g(s,T) - g(s,0).$$

To shorten the notation, we customarily write

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \{g(s,t)\}_0^\infty ,$$

understanding that the limit should be taken as $t \to \infty$. Appendix E contains a brief discussion of improper integrals of this type for readers who are meeting them for the first time, and for readers who would like a quick review.

For our purposes, s is a real variable, in which case F is a real-valued function of a real variable s. The reader should be aware, however, that in advanced applications of Laplace transforms, especially for solving partial differential equations and in many areas of electrical engineering, s is complex, in which case F is a complex-valued function of a complex variable.

We should determine properties of a function that guarantee existence of its Laplace transform. The following three examples point us in the correct direction.

Example 6.1 Find the Laplace transform of $f(t) = e^{at}$ where $a \neq 0$ is a constant.

Solution According to equation 6.2, the Laplace transform has values defined by

$$F(s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{(a-s)t}\, dt = \left\{ \frac{1}{a-s} e^{(a-s)t} \right\}_0^\infty = \frac{1}{a-s} \left[\lim_{t\to\infty} e^{(a-s)t} - 1 \right].$$

This limit exists, and has value 0, only when $s > a$. Hence, the Laplace transform of $f(t) = e^{at}$ is $1/(s-a)$, but only for $s > a$. We write

$$F(s) = \frac{1}{s-a}, \qquad s > a. \bullet$$

Example 6.2 Find the Laplace transform of $f(t) = t$.

Solution According to equation 6.2, the Laplace transform has values defined by

$$F(s) = \int_0^\infty t\, e^{-st}\, dt.$$

Integration by parts leads to

$$F(s) = \left\{ -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right\}_0^\infty = \lim_{t\to\infty} \left(-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right) + \frac{1}{s^2}.$$

This limit exists, and has value 0, only when $s > 0$. In other words, the Laplace transform of $f(t) = t$ is $F(s) = 1/s^2$, but the function is only defined for $s > 0$.$\bullet$

Example 6.3 Find the Laplace transform of the discontinuous function $f(t) = \begin{cases} 2t^2, & 0 \le t \le 1 \\ 1, & t > 1. \end{cases}$

It is shown in Figure 6.1.

Solution According to equation 6.2, the Laplace transform has values defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 2t^2 e^{-st}\,dt + \int_1^\infty e^{-st}\,dt.$$

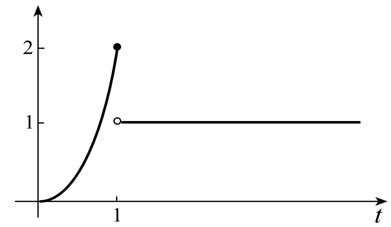


Figure 6.1

Two integrations by parts on the first integral lead to

$$F(s) = 2\left\{\left(-\frac{t^2}{s} - \frac{2t}{s^2} - \frac{2}{s^3}\right)e^{-st}\right\}_0^1 + \left\{\frac{-e^{-st}}{s}\right\}_1^\infty = -\left(\frac{1}{s} + \frac{4}{s^2} + \frac{4}{s^3}\right)e^{-s} + \frac{4}{s^3},$$

provided $s > 0$ •

What have we learned from Definition 6.1 and these three examples? First, when $f(t)$ is discontinuous, we subdivide the interval $0 < t < \infty$ into subintervals in which $f(t)$ is continuous. To avoid an infinite number of such subintervals, we could demand that $f(t)$ have a finite number of discontinuities. It turns out that this is not entirely necessary, although it is often the case. Instead, we demand that $f(t)$ have a finite number of discontinuities on every interval $0 \le t \le T$ of finite length. This would allow a periodic function like that in Figure 6.2 into our discussions. Such a function could represent an applied voltage in an electrical circuit that was periodically turned off. In addition, to guarantee existence of the integral of $e^{-st}f(t)$ on each subinterval in which $f(t)$ is continuous, we demand that right- and left-hand limits of $f(t)$ exist at every discontinuity. When a function has a finite number of discontinuities on an interval and right- and left-hand limits exist at all discontinuities in the interval, the function is said to be **piecewise-continuous** on that interval. We shall assume therefore that $f(t)$ is piecewise continuous on every interval $0 \le t \le T$ of finite length.

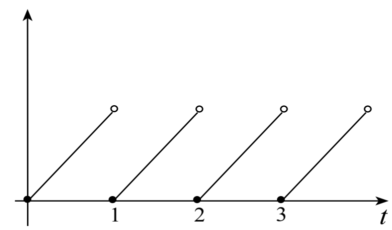


Figure 6.2

The second thing that we saw in Examples 6.1–6.3 is that there is always a restriction on values of s. The function $F(s)$ is not defined for all s; it is defined only for s larger than some number (a in Example 6.1 and 0 in Examples 6.2 and 6.3). This is due to the fact that for improper integral 6.2 to converge, the integrand must approach 0 as $t \to \infty$, and must do so sufficiently quickly. This means that $f(t)$ must not increase so rapidly that it cannot be suppressed by e^{-st} for some value of s. A sufficient restriction on the growth of $f(t)$ for large t is contained in the following definition.

Definition 6.2 A function $f(t)$ is said to be of **exponential order** α, written $O(e^{\alpha t})$, if there exist positive constants T and M such that $|f(t)| < Me^{\alpha t}$ for all $t > T$.

What this says algebraically is that for sufficiently large t ($t > T$), $|f(t)|$ must grow no faster than a constant M times $e^{\alpha t}$. Geometrically, the graph of $|f(t)|$ must be below that of $Me^{\alpha t}$ for $t > T$. It is important to realize that the exponential order of a function $f(t)$, if it has one, is concerned with function behaviour for very large t, not for small t. The absolute value $|f(t)|$ must eventually be less than $Me^{\alpha t}$, and stay less, but it need not be so for all t. This is shown in Figure 6.3. For example, the exponential function e^{4t} is $O(e^{4t})$ since M can be chosen as 2 and T as zero. Constant functions are of exponential order zero. The trigonometric functions $\sin at$ and $\cos at$ are $O(e^{0t})$ since both are less than $2 = 2e^{0t}$ for all t. The exponential order of t^n is discussed in the following example.

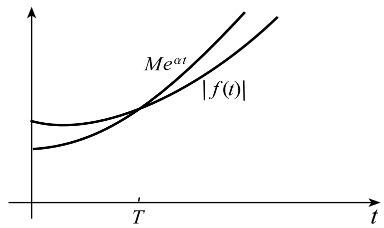


Figure 6.3

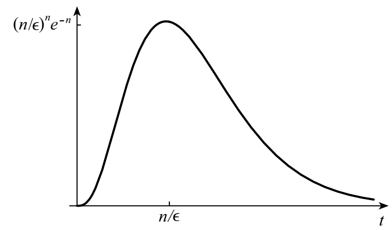


Figure 6.4

Example 6.4 Show that the function t^n, where n is a positive integer, is $O(e^{\epsilon t})$ for arbitrarily small, positive ϵ.

Solution Consider the function $f(t) = t^n e^{-\epsilon t}$ for arbitrary $\epsilon > 0$. To draw its graph we first calculate that

$$f'(t) = nt^{n-1}e^{-\epsilon t} - \epsilon t^n e^{-\epsilon t} = t^{n-1}e^{-\epsilon t}(n - \epsilon t).$$

There is a relative maximum at $t = n/\epsilon$ and when this is combined with the fact that $\lim_{t\to\infty} t^n e^{-\epsilon t} = 0$, the graph in Figure 6.4 results. It shows that the function $t^n e^{-\epsilon t}$ is bounded by $M = (n/\epsilon)^n e^{-n}$ for all $t \geq 0$. In other words, $t^n e^{-\epsilon t} < 2M$ for all $t > 0$; that is, $t^n < 2Me^{\epsilon t}$ for $t > 0$, and t^n is $O(e^{\epsilon t})$.●

We now show that piecewise-continuous functions of exponential order always have Laplace transforms.

Theorem 6.1 If $f(t)$ is piecewise-continuous on every finite interval $0 \leq t \leq T$, and is of exponential order α, then its Laplace transform exists for $s > \alpha$.

Proof The improper integral in equation 6.2 can be divided into integrals over the intervals $0 \leq t \leq T$ and $T \leq t < \infty$, for any T,

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt.$$

Since $f(t)$ is piecewise-continuous on $0 \leq t \leq T$, there is no question that the first of these integrals exists, and does so for all values of s. Furthermore, since $f(t)$ is $O(e^{\alpha t})$, there exist constants M and T such that $|f(t)| < Me^{\alpha t}$ for $t > T$. Hence,

$$\left|\int_T^\infty e^{-st} f(t)\,dt\right| \leq \int_T^\infty e^{-st}|f(t)|\,dt < \int_T^\infty Me^{-st}e^{\alpha t}\,dt = \int_T^\infty Me^{(\alpha-s)t}\,dt$$
$$= \left\{\frac{M}{\alpha - s}e^{(\alpha-s)t}\right\}_T^\infty = \frac{M}{s-\alpha}e^{(\alpha-s)T},$$

provided $s > \alpha$. In other words, the improper integral over the interval $T \leq t \leq \infty$ converges when $s > \alpha$. Thus, the Laplace transform of $f(t)$ is defined for $s > \alpha$. ∎

Theorem 6.1 provides sufficient conditions for existence of Laplace transforms. Functions that are not piecewise continuous or not of exponential order may or may not have transforms. For example, the function $f(t) = 1/\sqrt{t}$ is not piecewise continuous due to the infinite discontinuity at $t = 0$. It does, however, have a Laplace transform (see Exercise 35).

In calculating Laplace transforms of known functions by means of Definition 6.1, it is not necessary to determine whether the function is of exponential order prior to use of the integral; evaluation of the integral will yield the interval on which the transform is defined. When using techniques other than the defining integral to find Laplace transforms, however, it may be necessary to know that the function is of exponential order and piecewise-continuous on every finite interval. We shall develop other techniques in the next section. In this section we concentrate on the integral definition for the transform.

Example 6.5 Find the Laplace transform for $f(t) = t^n$, where n is a positive integer.

Solution Integration by parts gives

$$F(s) = \int_0^\infty t^n e^{-st}\,dt = \left\{\frac{t^n e^{-st}}{-s}\right\}_0^\infty - \int_0^\infty -\frac{n}{s}t^{n-1}e^{-st}\,dt = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt,$$

provided $s > 0$. A second integration by parts yields

$$\begin{aligned} F(s) = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt &= \frac{n}{s}\left\{\frac{t^{n-1}e^{-st}}{-s}\right\}_0^\infty - \frac{n}{s}\int_0^\infty -\frac{n-1}{s}t^{n-2}e^{-st}\,dt \\ &= \frac{n(n-1)}{s^2}\int_0^\infty t^{n-2}e^{-st}\,dt. \end{aligned}$$

Further intergations by parts lead to

$$F(s) = \frac{n(n-1)(n-2)\cdots(1)}{s^n}\int_0^\infty e^{-st}\,dt = \frac{n!}{s^n}\left\{\frac{e^{-st}}{-s}\right\}_0^\infty = \frac{n!}{s^{n+1}},$$

provided again that $s > 0$. This is consistent with Theorem 6.1 and Example 6.4. According to Example 6.4, t^n is $O(e^{\epsilon t})$ for arbitrarily small, positive ϵ, and therefore its Laplace transform should exist for $s > \epsilon$ for arbitrarily small $\epsilon > 0$. This is tantamount to $s > 0$.●

Example 6.6 Find the Laplace transform for $f(t) = \cos at$, where $a > 0$ is a constant.

Solution In Exercise 33, you are asked to perform two integrations by parts on the integral

$$\int_0^\infty e^{-st}\cos at\,dt$$

in order to find the Laplace transform. We provide an alternative using complex exponentials, and we do so for two reasons. First, the use of complex exponentials provides a much easier derivation; in particular, it avoids integration by parts. Secondly, complex numbers can facilitate many calculations in this chapter, and the sooner you are exposed to them the better. Instead of evaluating the above integral, we consider the integral

$$\int_0^\infty e^{-st}e^{ati}dt.$$

We have simply replaced $\cos at$ with e^{ati}, remembering that by Euler's identity, $e^{ati} = \cos at + i\sin at$. Now,

$$\int_0^\infty e^{-st}e^{ati}dt = \int_0^\infty e^{(-s+ai)t}dt = \left\{\frac{e^{(-s+ai)t}}{-s+ai}\right\}_0^\infty.$$

The limit of the antiderivative as $t \to \infty$ is

$$\lim_{t\to\infty}\left(\frac{e^{-st}e^{ati}}{-s+ai}\right) = \lim_{t\to\infty}\left[\frac{e^{-st}(\cos at + i\sin at)}{-s+ai}\right] = 0,$$

provided $s > 0$. Thus,

$$\int_0^\infty e^{-st}e^{ati}dt = \frac{1}{s-ai} = \frac{s+ai}{s^2+a^2}.$$

If we write the integral in the form

$$\int_0^\infty e^{-st}(\cos at + i\sin at)\,dt = \frac{s+ai}{s^2+a^2},$$

and take real and imaginary parts, we get

$$\int_0^\infty e^{-st}\cos at\,dt = \frac{s}{s^2+a^2}, \qquad \int_0^\infty e^{-st}\sin at\,dt = \frac{a}{s^2+a^2}.$$

In other words,

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}.$$

Thus, by using complex exponentials, we not only found the Laplace transform of $\cos at$, but we also found the transform of $\sin at$.●

The following table contains Laplace transforms of functions that occur very frequently in differential equations. They can be verified with equation 6.2.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$	e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$	$\cos at$	$\frac{s}{s^2+a^2}$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sin at - at\cos at$	$\frac{2a^3}{(s^2+a^2)^2}$	$at\cos at + \sin at$	$\frac{2as^2}{(s^2+a^2)^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$

Table 6.1

We have included transforms of the hyperbolic sine and cosine functions, but we make no use of them in this chapter. We do this for the sake of those who have not studied hyperbolic functions. Those familiar with these functions will be able to provide simpler solutions to some of the examples and exercises.

The Inverse Laplace Transform

Definition 6.3 When F is the Laplace transform of f, we call f the inverse Laplace transform of F, and write

$$f = \mathcal{L}^{-1}\{F\}. \tag{6.3}$$

For instance, Table 6.1 yields

$$\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+3}\right\} = \cos\sqrt{3}t.$$

The Laplace transform $F(s)$ of a function $f(t)$ is unique, every function has exactly one Laplace transform. On the other hand, many functions have the same transform. For example, the functions

$$f(t) = t^2 \qquad \text{and} \qquad g(t) = \begin{cases} 0, & t = 1 \\ t^2, & t \neq 1, 2 \\ 0, & t = 2, \end{cases}$$

which are identical except for their values at $t = 1$ and $t = 2$ both have the same transform $2/s^3$. The fact that $F(s) = 2/s^3$ follows from Table 6.1; $G(s) = 2/s^3$ follows from integration, (or by noting that because $f(t)$ and $g(t)$ differ only at isolated points, this makes no difference to integral 6.2). What we are saying is that the inverse transform $f = \mathcal{L}^{-1}\{F\}$ in Definition 6.3 is not an inverse in the true sense of inverse; there are many possibilities for f for given F. In advanced work, a formula for calculating inverse transforms is derived, and this formula always yields a continuous function $f(t)$, when this is possible. In the event that this is not possible, the formula gives a piecewise-continuous function whose value is the average of right- and left-limits at discontinuities, namely $\lim_{\epsilon\to 0}\,[f(t+\epsilon) + f(t-\epsilon)]/2$. The importance of this formula is that it defines $f = \mathcal{L}^{-1}\{F\}$ in a unique way. Other functions which have the same transform F differ from f only in their values at isolated points; they cannot differ from f over an entire interval $a \le t \le b$. When f is a continuous function with transform F, there cannot be another continuous function with the same transform. With this in mind, we adopt the procedure of choosing a continuous function $\mathcal{L}^{-1}\{F\}$ for given F whenever this is possible. In the confines of differential equations, this will always be possible since solutions of differential equations are always continuous functions. Thus, when solving differential equations by Laplace transforms, there will always be only one inverse transform for a given function $F(s)$. In other words, we can talk about **the** inverse Laplace transform of $F(s)$.

According to the following theorem, the Laplace transform and its inverse are linear operators in the sense of equation 4.12. That the transform is linear is a direct result of the fact that integration is a linear operation; once the transform is linear, so also is the inverse.

Theorem 6.2 The Laplace transform and its inverse are linear operators; that is, for arbitrary functions f and g, arbitrary transforms F and G, and an arbitrary constant c,

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}, \qquad \mathcal{L}\{c\,f\} = c\,[\mathcal{L}\{f\}], \tag{6.4a}$$

$$\mathcal{L}^{-1}\{F+G\} = \mathcal{L}^{-1}\{F\} + \mathcal{L}^{-1}\{G\}, \qquad \mathcal{L}^{-1}\{c\,F\} = c\,[\mathcal{L}^{-1}\{F\}]. \tag{6.4b}$$

For instance, using linearity and Table 6.1,

$$\mathcal{L}\{2e^{-t} + 3\sin 4t\} = 2\mathcal{L}\{e^{-t}\} + 3\mathcal{L}\{\sin 4t\} = \frac{2}{s+1} + 3\left(\frac{4}{s^2+16}\right),$$

and

$$\mathcal{L}^{-1}\left\{\frac{2}{s^4} - \frac{4s}{s^2+5}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} - 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\} = 2\left(\frac{t^3}{6}\right) - 4\cos\sqrt{5}t.$$

The following result can serve as a partial check on calculations of Laplace transforms.

Theorem 6.3 If $f(t)$ is is piecewise-continuous on every finite interval $0 \le t \le T$, and is of exponential order α, its Laplace transform has limit zero as $s \to \infty$; that is,

$$\lim_{s\to\infty} F(s) = 0. \tag{6.5}$$

Proof: The definition of $F(s)$ gives

$$|F(s)| = \left|\int_0^\infty e^{-st} f(t)\,dt\right| \le \int_0^\infty e^{-st}|f(t)|\,dt = \int_0^T e^{-st}|f(t)|dt + \int_T^\infty e^{-st}|f(t)|dt.$$

Since $f(t)$ is piecewise continuous on $0 \le t \le T$, it is bounded thereon, and there exists a number $\overline{M}$ such that $|f(t)| < \overline{M}$ in this interval. Furthermore, since $f(t)$ is of exponential order α, there exist constants $T > 0$ and $M > 0$, such that for $t > T$, $|f(t)| < Me^{\alpha t}$. We can therefore write that

$$\begin{aligned}|F(s)| &< \int_0^T \overline{M}e^{-st}dt + \int_T^\infty e^{-st}Me^{\alpha t}dt = \overline{M}\left\{\frac{e^{-st}}{-s}\right\}_0^T + M\left\{\frac{e^{(\alpha-s)t}}{\alpha - s}\right\}_T^\infty \\ &= \frac{\overline{M}}{s}(1 - e^{-sT}) + \frac{Me^{(\alpha-s)T}}{s-\alpha},\end{aligned}$$

provided that $s > 0$ and $s > \alpha$. The limit of this is zero as $s \to \infty$. ∎

We said that this theorem could serve as a check on calculations. For instance, if we calculated the transform of a function $f(t)$ (piecewise continuous on every finite interval, and of exponential order) to be

$$F(s) = \frac{s^2 + 2s - 5}{3s^2 + 10s + 15},$$

we would know that we had made an error since the limit of this function as $s \to \infty$ is not equal to zero; it is equal to 1/3. In Section 6.5, we will encounter a function that is an exception to this rule, but it will not be a function that is piecewise continuous. Another check on the validity of a Laplace transform can be found in Exercise 52 of Section 6.3.

We now give you a preview of what is to come. Consider solving the initial-value problem

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = 6 - 5t, \qquad y(0) = 1, \quad y'(0) = -1. \tag{6.6}$$

We can certainly solve this with the techniques of Chapter 4. Use the auxiliary equation to find a general solution $y_h(t)$ of the associated homogeneous equation;

use undetermined coefficients to find a particular solution $y_p(t)$ of the differential equation; add these together and evaluate constants with the initial conditions. The Laplace transform takes an entirely different approach; it reduces the initial-value problem to an algebraic equation. To do this, we need to know how to take Laplace transforms of derivatives. We quote two results that will be verified in Section 6.3. If $F(s)$ is the Laplace transform of $f(t)$, then Laplace transforms of $f'(t)$ and $f''(t)$ can be written in terms of $F(s)$ as follows:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \tag{6.7a}$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0). \tag{6.7b}$$

If we take Laplace transforms of both sides of differential equation 6.6, and use the fact that the transform is a linear operator, we obtain

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = 6\mathcal{L}\{1\} - 5\mathcal{L}\{t\}.$$

When we denote the Laplace transform of $y(t)$ by $Y(s)$, and use formulas 6.7 on the left, we get

$$[s^2Y(s) - s(1) - (-1)] - 4[sY(s) - 1] - 5Y(s) = \frac{6}{s} - \frac{5}{s^2}.$$

This is an algebraic equation for $Y(s)$ which is easily solved,

$$Y(s) = \frac{\dfrac{6}{s} - \dfrac{5}{s^2} + s - 5}{s^2 - 4s - 5}.$$

When this rational expression is simplified and written in its partial fraction decomposition, the result is

$$Y(s) = \frac{17/6}{s+1} + \frac{1/6}{s-5} + \frac{1}{s^2} - \frac{2}{s}.$$

We can take the inverse Laplace transform of each of these terms to find the solution of the initial-value problem,

$$y(t) = \frac{17}{6}e^{-t} + \frac{1}{6}e^{5t} + t - 2.$$

This example is typical of Laplace transforms at work on initial-value problems like those in Chapters 4 and 5. The transform reduces the differential equation in $y(t)$ to an algebraic equation in its transform $Y(s)$. The algebraic equation is solved for $Y(s)$ and the inverse transform then yields the solution $y(t)$ of the initial-value problem. In order to solve other initial-value problems, we need to expand the catalogue of functions with known Laplace transforms beyond those in Table 6.1. Furthermore, in Chapter 4 we assumed continuity of nonhomogeneous terms in linear differential equations. This was a matter of convenience rather than necessity. However, in Exercises 32 and 33 of Section 4.5, we hinted at the awkwardness of incorporating discontinuities into the techniques of Chapter 4. We shall give other examples of discontinuous nonhomogeneities in this chapter, and see how easily they are handled by Laplace transforms. Section 6.2 concentrates on efficient ways to calculate transforms and inverse transforms, and Sections 6.3 and 6.4 then return to full discussions of differential equations.

EXERCISES 6.1

In Exercises 1–10 use linearity and Table 6.1 to find the Laplace transform of the function.

1. $f(t) = t^3 - 2t^2 + 1$

2. $f(t) = t + e^t$

3. $f(t) = 5e^{4t}$

4. $f(t) = e^{-2t} + 2e^t$

5. $f(t) = \sin 4t + 3\cos 4t$

6. $f(t) = \cos 2t - 3\sin 4t$

7. $f(t) = 5t\,\cos 2t$

8. $f(t) = 3t\,\sin 4t$

9. $f(t) = 5t\,\cos t - 2t\,\sin t$

10. $f(t) = 3t\,\sin t - \cos t$

In Exercises 11–20 use linearity and Table 6.1 to find the inverse Laplace transform of the function.

11. $F(s) = \dfrac{7}{s^3}$

12. $F(s) = \dfrac{2}{s} - \dfrac{3}{s^4}$

13. $F(s) = \dfrac{1}{s+5} + \dfrac{4}{s^2}$

14. $F(s) = \dfrac{3}{s-1}$

15. $F(s) = \dfrac{s}{s^2+4} - \dfrac{3}{s^2+4}$

16. $F(s) = \dfrac{2s}{s^2+2} - \dfrac{5}{s^2+9}$

17. $F(s) = \dfrac{2s}{(s^2+2)^2}$

18. $F(s) = \dfrac{s^2}{(s^2+9)^2}$

19. $F(s) = \dfrac{3s - s^2}{(s^2+4)^2}$

20. $F(s) = \dfrac{s^2 - 2}{(s^2+3)^2}$

In Exercises 21–32 use Definition 6.1 to find the Laplace transform of the function.

21. $f(t) = \begin{cases} 0, & 0 < t < 3 \\ 1, & t > 3 \end{cases}$

22. $f(t) = \begin{cases} 1, & 0 < t < 4 \\ 2, & t > 4 \end{cases}$

23. $f(t) = \begin{cases} t, & 0 < t < 2 \\ 2, & t > 2 \end{cases}$

24. $f(t) = \begin{cases} t^2, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

25. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t^2, & t > 1 \end{cases}$

26. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ (t-1)^2, & t > 1 \end{cases}$

27. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

28. $f(t) = \begin{cases} t, & 0 < t < 1 \\ 2-t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

29. $f(t) = \begin{cases} 2t, & 0 < t < 1 \\ t, & t > 1 \end{cases}$

30. $f(t) = \begin{cases} 1+t^2, & 0 < t < 1 \\ 2t, & t > 1 \end{cases}$

31. $f(t) = \begin{cases} 0, & 0 < t < a \\ 1, & t > a \end{cases}$ (a constant)

32. $f(t) = \begin{cases} 0, & 0 < t < a \\ 1, & a < t < b \\ 0, & t > b \end{cases}$ (a, b constants)

33. Use integration by parts to find Laplace transforms for $\sin at$ and $\cos at$.

34. Derive the Laplace transforms for $t\cos at$ and $t\sin at$ in Table 6.1.

35. The function $1/\sqrt{t}$ is not piecewise continuous because of its infinite discontinuity at $t = 0$. Show that it has Laplace transform $\sqrt{\pi/s}$. Hint: Set $u = \sqrt{t}$ in the definition of the Laplace transform of $1/\sqrt{t}$ in terms of a definite integral and use the fact that $\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}/2$.

36. Are all bounded functions (functions that satisfy $|f(t)| < M$ for all $t > 0$) of exponential order?

37. Are all continuous functions of exponential order?

38. In Example 6.6 we used complex exponentials to find the Laplace transforms for $\cos at$ and $\sin at$. In Exercise 33, we used integration by parts. Another method that also works for other functions is to use Maclaurin series. The Maclaurin series for $\sin at$ is

$$\sin at = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(at)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)!} t^{2n+1}.$$

Assuming that the operation of taking the Laplace transform can be interchanged with the summation operation, derive the Laplace transform of $\sin at$.

39. Use the technique of Exercise 38 to derive the Laplace transform of $\cos at$.

40. The Bessel function of the first kind of order zero has Maclaurin series

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} t^{2n}.$$

Show that its Laplace transform is $\mathcal{L}\{J_0(t)\} = \dfrac{1}{\sqrt{1+s^2}}$.

41. Verify that $\mathcal{L}\left\{\dfrac{\sin at}{t}\right\} = \mathrm{Tan}^{-1}\left(\dfrac{a}{s}\right)$.

42. Verify that $\mathcal{L}\left\{\dfrac{e^{bt}-e^{at}}{t}\right\} = \ln\left(\dfrac{s-a}{s-b}\right)$ when $s > b > a$.

43. In Example 6.5, we developed the formula for the Laplace transform of t^n when n is a positive integer. In this exercise, we extend the result to the case that n is positive, but not an integer. First, we need to extend the definition of factorials by defining the gamma function,

$$\Gamma(t) = \int_0^{\infty} e^{-u} u^{t-1}\, du, \quad t > 0.$$

This function is discussed in more detail in Appendix A, but for our present purposes, only the property in part (a) is required.

(a) Verify that the gamma function satisfies the recursive formula

$$\Gamma(t+1) = t\,\Gamma(t), \qquad \text{and hence that} \qquad \Gamma(n+1) = n! \quad \text{when } n \text{ is positive integer.}$$

(b) Now prove that for $r > 0$,

$$\mathcal{L}\{t^r\} = \frac{\Gamma(r+1)}{s^{r+1}}.$$

44. (a) Prove that the function e^{t^2} is not of exponential order.

45. (a) Prove that the function $f(t) = \sin\left(e^{t^2}\right)$ is of exponential order.
(b) Prove that the derivative $f'(t)$ is not of exponential order? In spite of this, $f'(t)$ has a Laplace transform. (See Exercise 54 in Section 6.3.)

6.2 Algebraic Properties of The Laplace Transform and its Inverse

Early in your calculus studies you were required to use the limit-definition of the derivative to differentiate various functions. You were quickly brought to the realization that rules could be developed so that use of the definition could be eliminated, and you certainly appreciated these rules (power, product, and quotient to name a few). The same was true when it came to the definition of the definite integral as the limit of a summation. It is impossible to use the definition to find the definite integral of all but a handful of functions, and therefore using anti-derivatives to calculate definite integrals is essential. Likewise, seldom is it necessary to evaluate the improper integral in Definition 6.1 to find the Laplace transform for a function; other techniques prove more efficient. The purpose of this section is to develop some of these shortcuts. In addition, recall that our intention is to use Laplace transforms to provide another method for solving linear differential equations and extensions which are difficult or impossible to solve with the techniques of Chapter 4. With this in mind, note how many of the algebraic properties of the Laplace transform uncovered in this section are directed toward the functions so prevalent in solving linear differential equations, namely, t^n, e^{at}, $\sin at$, $\cos at$, and sums and products of these functions. In Section 6.3, we derive formulas 6.7 for taking Laplace transforms of derivatives of functions. With these formulas and the results of this section, we will be well prepared to solve differential equations.

One of two *shifting* properties is contained in the following theorem.

Theorem 6.4 When F is the Laplace transform of f,

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \tag{6.8a}$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t). \tag{6.8b}$$

Proof By Definition 6.1,

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{at}e^{-st}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt.$$

But this is equation 6.2 with s replaced by $s-a$; that is,

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

Equation 6.8b is 6.8a written in terms of inverse transforms rather than transforms. ∎

The notation in equation 6.8 is not quite as described earlier. The Laplace transform and its inverse operate on functions, not on function values as is suggested by 6.8. These equations would be more properly stated in the form

$$\mathcal{L}\{e^{at}f\}(s) = F(s-a), \tag{6.9a}$$

$$\mathcal{L}^{-1}\{F(s-a)\}(t) = e^{at}f(t). \tag{6.9b}$$

We feel that the shifting property is more clearly conveyed for most readers by 6.8a,b, and we apologize to readers who are offended by the notation. It may be convenient to repeat this practice in describing other properties of the Laplace transform, but we shall attempt to minimize its use.

When calculating transforms and inverse transforms, we often write properties 6.8 in the form

$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{|s\to s-a}, \tag{6.10a}$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}. \tag{6.10b}$$

On the right of property 6.10a, $\mathcal{L}\{f(t)\}$ is a function of s. The subscript $|s \to s-a$ means that each s is to be replaced by $s-a$.

Equation 6.8a, or its alternatives, states that multiplication by an exponential e^{at} in the t-domain is equivalent to a translation or shift by a in the s-domain. It provides a quick way to find the Laplace transform of any function $f(t)$ multiplied by an exponential, provided the Laplace transform of $f(t)$ is known. For example, property 6.10a implies that

$$\mathcal{L}\left\{t^2e^{-5t}\right\} = \mathcal{L}\left\{t^2\right\}_{|s\to s+5} = \left(\frac{2}{s^3}\right)_{|s\to s+5} = \frac{2}{(s+5)^3}.$$

Property 6.10b yields

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-6)^5}\right\} = e^{6t}\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = e^{6t}\left(\frac{t^4}{4!}\right).$$

Example 6.7 In Exercise 34 of Section 6.1, we found Laplace transforms for $t\cos at$ and $t\sin at$, by using Euler's identity and integration. Property 6.8a is superior if we again replace trigonometric functions with complex exponentials. This is an ongoing theme; always consider replacing sines and cosines with complex exponentials. We can write that

$$\mathcal{L}\{te^{ati}\} = \mathcal{L}\{t\}_{|s\to s-ai} = \left(\frac{1}{s^2}\right)_{|s\to s-ai} = \frac{1}{(s-ai)^2}.$$

We now display the real and imaginary parts of both sides of the equation,

$$\mathcal{L}\{t(\cos at + i\sin at)\} = \frac{(s+ai)^2}{(s-ai)^2(s+ai)^2} = \frac{(s^2-a^2)+2asi}{(s^2+a^2)^2}.$$

When we take real and imaginary parts,

$$\mathcal{L}\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}, \qquad \mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}.\bullet$$

Property 6.10b is particularly useful in finding inverse Laplace transforms of rational functions of s that contain irreducible quadratic factors in denominators. We encounter them constantly. Here is an example.

Example 6.8 Find the inverse Laplace transform for $F(s) = (s+1)/(s^2-6s+14)$.

Solution First, by completing the square on the quadratic, we can express $F(s)$ in the form

$$F(s) = \frac{s+1}{(s-3)^2+5} = \frac{(s-3)+4}{(s-3)^2+5}.$$

We can now use property 6.10b to find the inverse transform,

$$f(t) = e^{3t}\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+5}\right\} = e^{3t}\left(\cos\sqrt{5}t + \frac{4}{\sqrt{5}}\sin\sqrt{5}t\right).\bullet$$

In physical applications, we often encounter quantities that are turned on and off, or quantities that change abruptly. For example, in mixing problems, such as Example 3.10 of Section 3.4, the concentration of salt added to the tank could suddenly be changed at any time; the applied voltage in an LCR-circuit could be turned on and off any number of times, and the forcing function in a mass-spring system could be turned on or off, or sharply changed. Such functions are conveniently described by **Heaviside unit step functions** introduced in Section 5.5. The fundamental unit step function is

$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases} \tag{6.11}$$

(Some authors replace $t \geq 0$ in this definition with $t > 0$ so that the function is undefined at $t = 0$. The rest of this chapter can be developed with either convention with minor adjustments in results.) A graph of this function is shown in Figure 6.5; there is a discontinuity of magnitude unity at $t = 0$, hence the name unit step function.

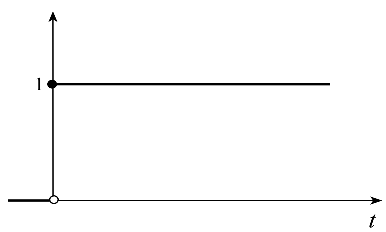


Figure 6.5

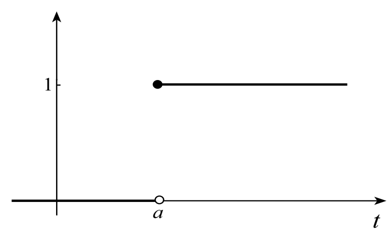


Figure 6.6

When the discontinuity occurs at $t = a$, the function is denoted by

$$h(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases} \tag{6.12}$$

Its graph is shown in Figure 6.6. This notation is consistent with that in elementary calculus where replacing the variable t in a function $f(t)$ by $t-a$ translates the graph of the function a units to the right.

Heaviside unit step functions provide compact representations for functions whose descriptions vary from one interval to another; such functions may have, or may not have, discontinuities at points that separate these intervals. Such a function is shown in Figure 6.7. It is defined differently on the intervals $0 < t \leq a$, $a < t < b$, and $t > b$. It is continuous at $t = a$, but not at $t = b$.

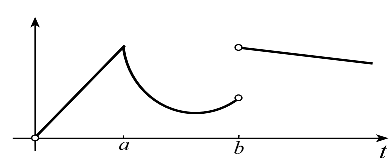


Figure 6.7

An important function in our discussions is shown in Figure 6.8. It is called a **pulse** function. It can be expressed algebraically in the form $h(t-a) - h(t-b)$, except at $t = a$ and $t = b$. In the event that the height of the nonzero portion is

c rather than unity (Figure 6.9), we obtain $c[h(t-a) - h(t-b)]$, again except at $t = a$ and $t = b$.

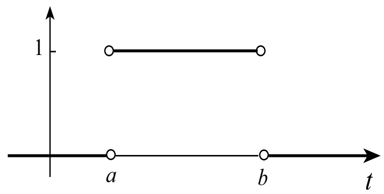


Figure 6.8

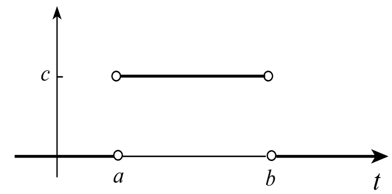


Figure 6.9

Pulse functions can be combined algebraically to describe *step functions*, such as that in Figure 6.10. It is the sum of two pulse functions,

$$4[h(t) - h(t-3)] + 2[h(t-3) - h(t-6)] = 4h(t) - 2h(t-3) - 2h(t-6),$$

except at $t = 0, 3$, and 6. The step function in Figure 6.11 is the sum of three pulses,

$$\begin{aligned} 3[h(t-a) - h(t-b)] + 4[h(t-b) - h(t-c)] + h(t-c) \\ = 3h(t-a) + h(t-b) - 3h(t-c), \end{aligned}$$

except at $x = a, b$, and c. In future representations of piecewise defined functions in terms of Heaviside functions, we will omit mentioning the exceptions.

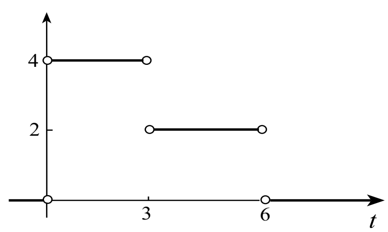


Figure 6.10

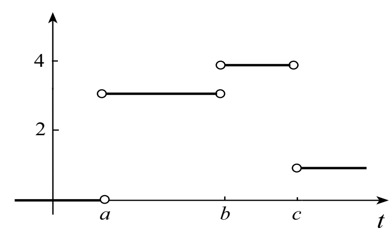


Figure 6.11

A convenient representation for the function in Figure 6.12 is $t^2[h(t) - h(t-a)]$, and for the function in Figure 6.13, $[2 - (t-a)/(b-a)][h(t-a) - h(t-b)]$.

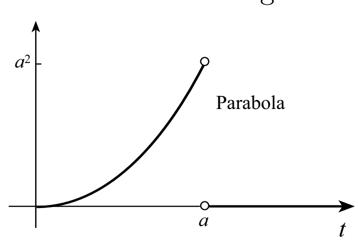


Figure 6.12

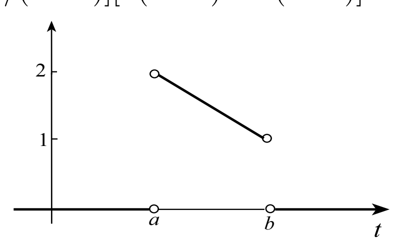


Figure 6.13

What these examples illustrate is that to "turn a function on" for $t \geq a$, multiply it by $h(t-a)$. It will be zero for $t < a$. To turn it on between $t = a$ and $t = b$, multiply it by $h(t-a) - h(t-b)$. It will be zero for $t < a$ and $t \geq b$. The parabola in Figure 6.14 has equation $a^2 + (t-a)^2$ for $t > a$. To turn it on, we multiply by $h(t-a)$; that is, the function can be expressed in the form $[a^2 + (t-a)^2]h(t-a)$. For the function in Figure 6.15, we turn on the straight line $y = a - a(t-a)/(b-a)$ for $a < t < b$, and then the horizontal line $y = c$ for $t > b$,

$$[a - a(t-a)/(b-a)][h(t-a) - h(t-b)] + c\,h(t-b).$$

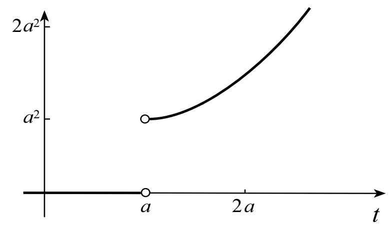


Figure 6.14

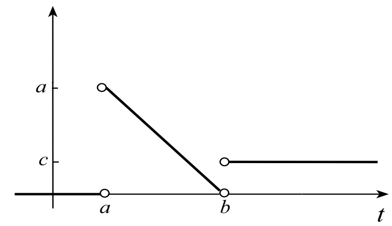


Figure 6.15

You may have noticed that in Figures 6.8–6.15, functions do not have values at discontinuities. Because of this, representations of these functions in terms of Heaviside functions are not valid at discontinuities. If a function has a value at a discontinuity, its representation in terms of Heaviside functions may or may not be valid at the discontinuity. For instance, the Heaviside representation of the functions in Figures 6.16a,b is

$$f(t) = \left[c + \left(\frac{b-c}{a}\right)(t-a)\right] h(t-a).$$

It is valid at the discontinuity $t = a$ in Figure 6.16a, but not in 6.16b. None of this really matters when it comes to Laplace transforms. Because the transform of a function is defined as a definite integral, the transform is the same whether the function has a value at a discontinuity or not. We will therefore continue the practice of leaving a function undefined at discontinuities (when the purpose is to take the transform of the function).

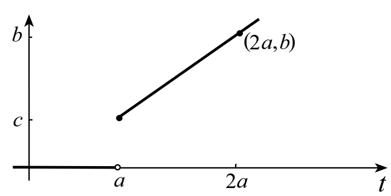


Figure 6.16a

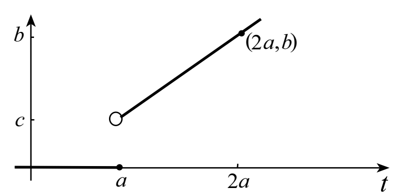


Figure 6.16b

The Laplace transform of the Heaviside unit step function is

$$\mathcal{L}\{h(t-a)\} = \int_0^\infty e^{-st} h(t-a)\, dt = \int_a^\infty e^{-st}\, dt = \left\{\frac{e^{-st}}{-s}\right\}_a^\infty = \frac{e^{-as}}{s}, \qquad (6.13)$$

provided $s > 0$.

In Section 6.1, we used integration to find the Laplace transform of piecewise defined functions. In the above discussions, we represented such functions as products of functions multiplied by Heaviside functions, and we did so in order to circumvent integrations. The following theorem enables us to do this.

Theorem 6.5 When $f(t)$ has a Laplace transform,

$$\mathcal{L}\{f(t)h(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}. \qquad (6.14a)$$

Proof According to Definition 6.1,

$$\mathcal{L}\{f(t)h(t-a)\} = \int_0^\infty e^{-st} f(t)h(t-a)\, dt = \int_a^\infty e^{-st} f(t)\, dt.$$

If we change variables of integration with $u = t - a$, then

$$\mathcal{L}\{f(t)h(t-a)\} = \int_0^\infty e^{-s(u+a)} f(u+a)\,du$$
$$= e^{-as}\int_0^\infty e^{-su} f(u+a)\,du = e^{-as}\mathcal{L}\{f(t+a)\}. \blacksquare$$

We illustrate how to use this result in the following examples.

Example 6.9 Find the Laplace transform for the function $f(t) = \begin{cases} 0, & 0 \le t \le 2 \\ (t-2)^2, & t > 2, \end{cases}$ shown in Figure 6.17.

Solution Since $f(t)$ can be expressed in the form $f(t) = (t-2)^2 h(t-2)$, equation 6.14a gives

$$F(s) = \mathcal{L}\{(t-2)^2 h(t-2)\} = e^{-2s}\mathcal{L}\{t^2\} = \frac{2e^{-2s}}{s^3}.\bullet$$

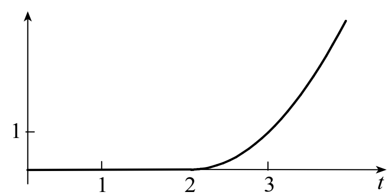


Figure 6.17

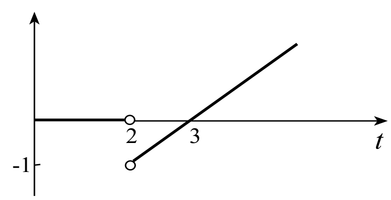


Figure 6.18

Example 6.10 Find the Laplace transform for the function $f(t) = \begin{cases} 0, & 0 \le t < 2 \\ t-3, & t > 2, \end{cases}$ shown in Figure 6.18.

Solution Since $f(t)$ can be expressed in the form $f(t) = (t-3)h(t-2)$, its Laplace transform is

$$F(s) = \mathcal{L}\{(t-3)h(t-2)\} = e^{-2s}\mathcal{L}\{(t+2)-3\} = e^{-2s}\mathcal{L}\{t-1\} = e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right).\bullet$$

Example 6.11 Find the Laplace transform for the function in Figure 6.19.

Solution The function is continuous, but because it is defined differently on the intervals $0 \le t \le 1$, $1 < t \le 2$, and $t > 2$, it can be represented efficiently in terms of Heaviside functions,

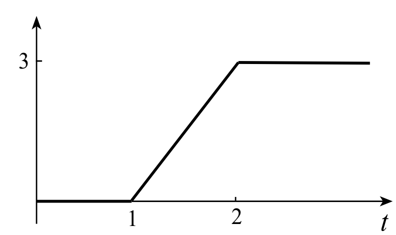


Figure 6.19

$$f(t) = 3(t-1)[h(t-1) - h(t-2)] + 3h(t-2)$$
$$= 3(t-1)\,h(t-1) + (6-3t)\,h(t-2).$$

We can now use equation 6.14a to find its Laplace transform,

$$F(s) = e^{-s}\mathcal{L}\{3t\} + e^{-2s}\mathcal{L}\{6 - 3(t+2)\} = \frac{3e^{-s}}{s^2} - e^{-2s}\left(\frac{3}{s^2}\right).\bullet$$

Example 6.12 Find the Laplace transform for $e^{-3t}\sin 2t\, h(t-1)$.

Solution Using property 6.14a,

$$\begin{aligned}\mathcal{L}\{e^{-3t}\sin 2t\, h(t-1)\} &= e^{-s}\mathcal{L}\{e^{-3(t+1)}\sin 2(t+1)\}\\ &= e^{-s-3}\mathcal{L}\{e^{-3t}\sin 2(t+1)\}\\ &= e^{-(s+3)}\mathcal{L}\{\sin 2(t+1)\}_{|s\to s+3} \qquad \text{(using equation 6.10a)}\\ &= e^{-(s+3)}\mathcal{L}\{\cos 2\,\sin 2t + \sin 2\,\cos 2t\}_{|s\to s+3}\\ &= e^{-(s+3)}\left[\frac{(\cos 2)2}{s^2+4} + \frac{(\sin 2)s}{s^2+4}\right]_{|s\to s+3}\\ &= e^{-(s+3)}\left[\frac{2\cos 2}{(s+3)^2+4} + \frac{(\sin 2)(s+3)}{(s+3)^2+4}\right].\bullet\end{aligned}$$

The equivalent of property 6.14a in terms of inverse transforms is equally as important as 6.14a itself, and from the inverse statement it gets its name the *second shifting property* of Laplace transforms. We state it as a corollary to Theorem 6.5.

Corollary 6.5.1 If $f = \mathcal{L}^{-1}\{F\}$, then

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)h(t-a) = \mathcal{L}^{-1}\{F(s)\}_{|t\to t-a}h(t-a). \qquad (6.14b)$$

The graph of $f(t-a)h(t-a)$ is that of $f(t)$ (Figure 6.20a) shifted a units to the right and turned on for $t > a$ (Figure 6.20b).

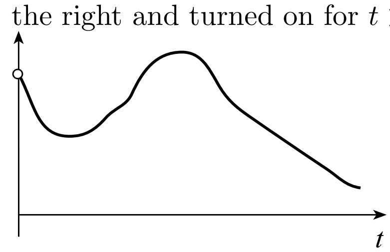


Figure 6.20a

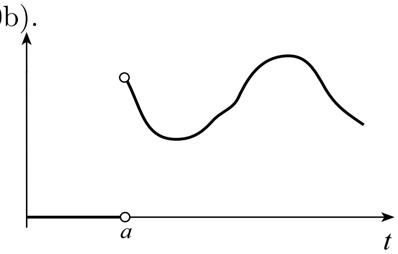


Figure 6.20b

Thus, to find the inverse transform of a function in the form $e^{-as}F(s)$, we find the inverse transform of $F(s)$, translate it a units to the right, and turn it on for $t > a$.

Example 6.13 Find $\mathcal{L}^{-1}\left\{\dfrac{5e^{-4s}}{s^3}\right\}$.

Solution With property 6.14b,

$$\mathcal{L}^{-1}\left\{\frac{5e^{-4s}}{s^3}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}_{|t\to t-4}h(t-4) = 5\left(\frac{t^2}{2}\right)_{|t\to t-4}h(t-4) = \frac{5}{2}(t-4)^2h(t-4).\bullet$$

A graph of $(5/2)(t-4)^2h(t-4)$ is shown in Figure 6.21.

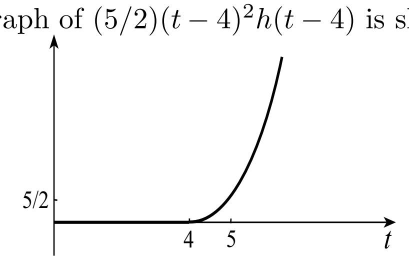


Figure 6.21

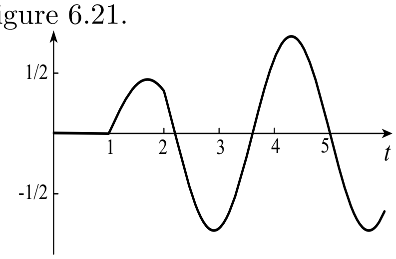


Figure 6.22

Example 6.14 Find the inverse transform for $F(s) = \dfrac{e^{-s} - e^{-2s}}{s^2+5}$.

Solution Property 6.14b gives

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+5}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+5}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2+5}\right\}_{|t\to t-1} h(t-1) - \mathcal{L}^{-1}\left\{\frac{1}{s^2+5}\right\}_{|t\to t-2} h(t-2)\\
&= \left(\frac{1}{\sqrt{5}}\sin\sqrt{5}t\right)_{|t\to t-1} h(t-1) - \left(\frac{1}{\sqrt{5}}\sin\sqrt{5}t\right)_{|t\to t-2} h(t-2)\\
&= \frac{1}{\sqrt{5}}\left[\sin\sqrt{5}(t-1)\,h(t-1) - \sin\sqrt{5}(t-2)\,h(t-2)\right].
\end{aligned}$$

The function is shown in Figure 6.22. We can write it without the Heaviside functions as follows:

$$f(t) = \begin{cases} 0, & 0 \le t \le 1\\ (1/\sqrt{5})\sin\sqrt{5}(t-1), & 1 < t \le 2\\ (1/\sqrt{5})[\sin\sqrt{5}(t-1) - \sin\sqrt{5}(t-2)], & t > 2.\bullet \end{cases}$$

Finding inverse transforms is often a matter of finding the partial fraction decomposition of a rational function, together with the above properties and a set of tables. We illustrate in the following example. For readers who have never studied partial fractions, or need a refresher on the topic, we have provided coverage of the topic in Appendix D.

Example 6.15 Find inverse Laplace transforms for the following functions:

(a) $F(s) = \dfrac{s^2 - 9s + 9}{s^3(s^2+9)}$ (b) $F(s) = \dfrac{1}{s^2(s^2-4)}$ (c) $F(s) = \dfrac{e^{-s}}{s^2 - s}$

Solution (a) The partial fraction decomposition of $F(s)$ gives

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s^2-9s+9}{s^3(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3} - \frac{1}{s^2} + \frac{1}{s^2+9}\right\} = \frac{t^2}{2} - t + \frac{1}{3}\sin 3t.$$

(b) Once again partial fractions give

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/16}{s-2} - \frac{1/16}{s+2} - \frac{1/4}{s^2}\right\} = \frac{1}{16}e^{2t} - \frac{1}{16}e^{-2t} - \frac{t}{4}.$$

(c) With partial fractions and property 6.14b,

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2-s}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{s-1} - \frac{1}{s}\right)\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s}\right\}_{|t\to t-1} h(t-1)\\
&= (e^t - 1)_{|t\to t-1} h(t-1) = (e^{t-1} - 1)h(t-1).\bullet
\end{aligned}$$

Periodic Functions

The sine and cosine functions are periodic and there was no difficulty in finding their transforms. The function in Figure 6.23a is also periodic, but it is not obvious how to find its transform. To use the definition of the transform as an improper integral would require the addition of an infinite number of definite integrals, one over each period of the function. Alternatively, we could write the periodic function as the sum of an infinite number of functions, turned on and off by Heaviside functions, and add all their transforms. Fortunately, neither of these procedures is necessary; we can develop a method for finding the Laplace transform of a periodic function that involves one integration over one period, or a procedure that involves no integration at all, and we can do this in two ways. We use the above ideas, but in a general format with an unspecified function. Suppose then that $f(t)$ is periodic with period p, such as that in Figure 6.23a.

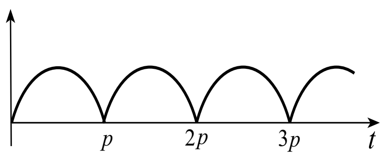


Figure 6.23a

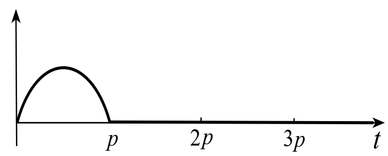


Figure 6.23b

We divide the range of integration in Definition 6.1 into two parts

$$F(s) = \int_0^p e^{-st} f(t)\,dt + \int_p^\infty e^{-st} f(t)\,dt.$$

We substitute $u = t - p$ in the second integral,

$$F(s) = \int_0^p e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+p)} f(u+p)\,du.$$

But $f(t)$ has period p, so that $f(u+p) = f(u)$, and therefore

$$F(s) = \int_0^p e^{-st} f(t)\,dt + e^{-ps} \int_0^\infty e^{-su} f(u)\,du = \int_0^p e^{-st} f(t)\,dt + e^{-ps} F(s).$$

We can solve this for

$$F(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\,dt. \tag{6.15}$$

The is the first of the results that we sought, a method for finding the transform of a periodic function that involves only integration over one period of the function. The second method is to not integrate at all. Suppose that $f_1(t)$ denotes the function that is equal to $f(t)$ over the first period $0 \le t \le p$ of the function, and is otherwise equal to zero (Figure 6.23b). This function is sometimes called the **windowed version** of $f(t)$. The Laplace transform of $f_1(t)$ is defined by the definite integral in equation 6.15. In other words, we can write that

$$F(s) = \frac{1}{1 - e^{-ps}} \mathcal{L}\{f_1(t)\}. \tag{6.16}$$

This is the second result that we were looking for, a method to calculate the Laplace transform of a periodic function that does not require integration. Formula 6.16 does this, provided we are willing, and able, to find $\mathcal{L}\{f_1(t)\}$ without integration.

The same results can be obtained in another way. We write the function $f_1(t)$ in terms of $f(t)$,

$$f_1(t) = f(t)[h(t) - h(t-p)] = f(t)[1 - h(t-p)].$$

We now take Laplace transforms, and use property 6.14a,

$$F_1(s) = F(s) - e^{-ps}\mathcal{L}\{f(t+p)\}.$$

But $f(t+p) = f(t)$, so that

$$F_1(s) = F(s) - e^{-ps}\mathcal{L}\{f(t)\} = F(s) - e^{-ps}F(s) = (1 - e^{-ps})F(s).$$

When we solve this equation for $F(s)$, we get

$$F(s) = \frac{1}{1-e^{-ps}}F_1(s) = \frac{1}{1-e^{-ps}}\mathcal{L}\{f_1(t)\},$$

equation 6.16.

Example 6.16 Find the Laplace transform for the periodic function in Figure 6.24a.

Solution Since the function has period 2, formula 6.15 yields

$$F(s) = \frac{1}{1-e^{-2s}}\int_0^2 (1-t)e^{-st}\,dt.$$

Integration by parts gives

$$F(s) = \frac{1}{1-e^{-2s}}\left\{\frac{(t-1)}{s}e^{-st} + \frac{1}{s^2}e^{-st}\right\}_0^2 = \frac{1+e^{-2s}}{s(1-e^{-2s})} - \frac{1}{s^2}.$$

Alternatively, formula 6.16 gives

$$F(s) = \frac{1}{1-e^{-ps}}\mathcal{L}\{f_1(t)\},$$

where $f_1(t)$ is the windowed version of $f(t)$ in Figure 6.24b. Its Laplace transform is

$$\begin{aligned}\mathcal{L}\{(1-t)[h(t)-h(t-2)]\} &= \mathcal{L}\{(1-t)\,h(t)\} + \mathcal{L}\{(t-1)\,h(t-2)\}\\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-2s}\mathcal{L}\{t+1\}\\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right).\end{aligned}$$

Hence,

$$F(s) = \frac{1}{1-e^{-2s}}\left[\frac{1}{s} - \frac{1}{s^2} + e^{-2s}\left(\frac{1}{s^2}+\frac{1}{s}\right)\right] = \frac{1+e^{-2s}}{s(1-e^{-2s})} - \frac{1}{s^2}. \bullet$$

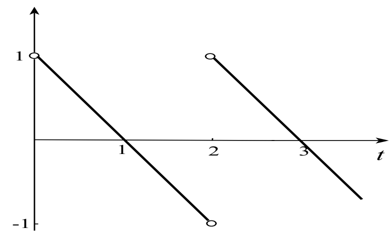


Figure 6.24a

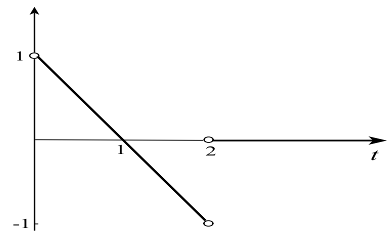


Figure 6.24b

Example 6.17 Find the Laplace transform for $|\sin 2t|$.

Solution Since $|\sin 2t|$ has period $\pi/2$ (see Figure 6.25), formula 6.15 gives

$$\mathcal{L}\{|\sin 2t|\} = \frac{1}{1-e^{-\pi s/2}}\int_0^{\pi/2} e^{-st}\sin 2t\,dt,$$

and we could use integration by parts to evaluate this integral. Alternatively, we can use formula 6.16,

$$\begin{aligned}\mathcal{L}\{|\sin 2t|\} &= \frac{1}{1-e^{-\pi s/2}}\mathcal{L}\{\sin 2t[h(t)-h(t-\pi/2)]\}\\ &= \frac{1}{1-e^{-\pi s/2}}\left[\mathcal{L}\{\sin 2t\} - \mathcal{L}\{\sin 2t\,h(t-\pi/2)\}\right]\\ &= \frac{1}{1-e^{-\pi s/2}}\left[\frac{2}{s^2+4} - e^{-\pi s/2}\mathcal{L}\{\sin 2(t+\pi/2)\}\right]\\ &= \frac{1}{1-e^{-\pi s/2}}\left[\frac{2}{s^2+4} + e^{-\pi s/2}\mathcal{L}\{\sin 2t\}\right]\\ &= \frac{1}{1-e^{-\pi s/2}}\left[\frac{2}{s^2+4} + \frac{2e^{-\pi s/2}}{s^2+4}\right]\\ &= \frac{2(1+e^{-\pi s/2})}{(s^2+4)(1-e^{-\pi s/2})}.\bullet\end{aligned}$$

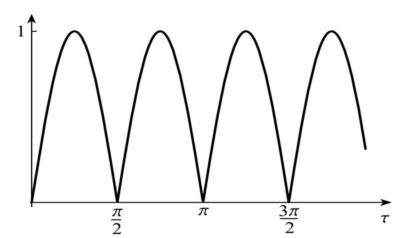


Figure 6.25

We have just seen that Laplace transforms of periodic functions that are not sinusoids contain factors $1/(1-e^{-ps})$. When we solve differential equations that have periodic inputs in Section 6.4, we will have to invert transforms with such factors. The following example illustrates how to do this.

Example 6.18 Find the inverse Laplace transform for $F(s) = \dfrac{2}{s^3(1-e^{-2s})}$.

Solution Property 6.14b enables us to find the inverse transform of any function multiplied by e^{-as}. We can write the above $F(s)$ as a sum of terms in this form if we expand $1/(1-e^{-2s})$ in a geometric seris

$$F(s) = \frac{2}{s^3(1-e^{-2s})} = \frac{2}{s^3}\left(1+e^{-2s}+e^{-4s}+e^{-6s}+\cdots\right).$$

We can now invert each term,

$$f(t) = t^2 + (t-2)^2 h(t-2) + (t-4)^2 h(t-4) + \cdots = \sum_{n=0}^{\infty} (t-2n)^2 h(t-2n).\bullet$$

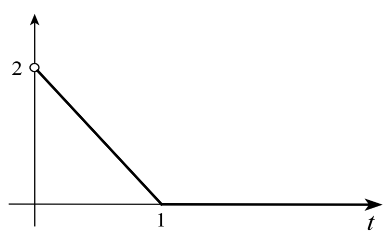
Properties of Laplace transforms and their inverses that we have discussed in this section have been gathered together for quick reference in Table 6.2. Included also are the transform pairs in Table 6.1. There are also properties that have yet to be considered, namely lines 12, 17, and 18. These will be developed in subsequent sections. Notice the arrows in the middle column. A double arrow $\leftrightarrow$ indicates that this line is useful in taking Laplace transforms and their inverses; a right arrow $\rightarrow$ indicates that the property is most useful in taking transforms; and a left arrow $\leftarrow$ indicates that the property is most useful in taking inverse transforms.

EXERCISES 6.2

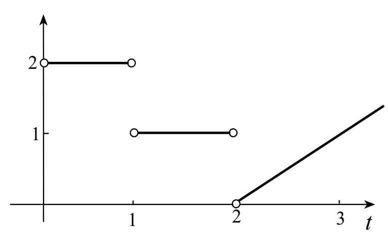
In Exercises 1–12 represent the functions in Exercises 21–32 of Section 6.1 in terms of Heaviside unit step functions. Find the Laplace transform of each function.

In Exercises 13–20 represent the function algebraically in terms of Heaviside unit step functions. Find the Laplace transform of each function.

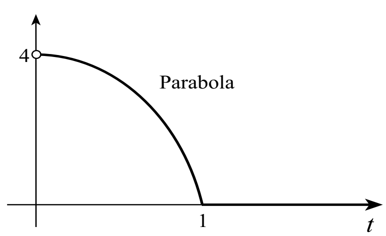
13.



14.



15.



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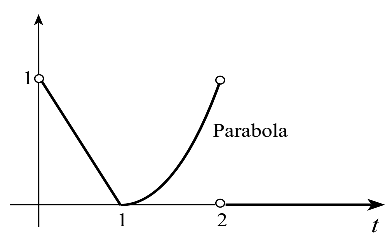
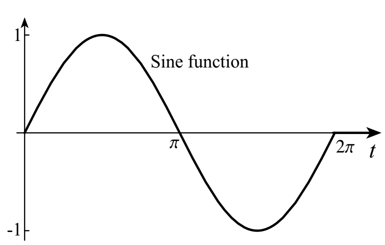
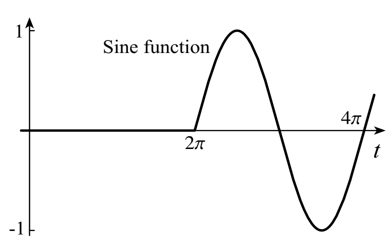


17.



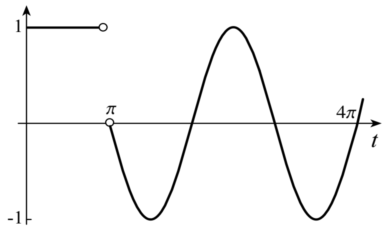
18.



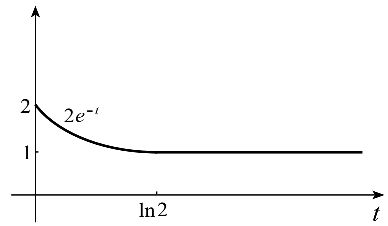
$f(t)$		$F(s)=\mathcal{L}\{f\}(s)$
$t^n \quad (n=0,1,2,\ldots)$	$\leftrightarrow$	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\leftrightarrow$	$\dfrac{1}{s-a}$
$\sin at$	$\leftrightarrow$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\leftrightarrow$	$\dfrac{s}{s^2+a^2}$
$t\sin at$	$\leftrightarrow$	$\dfrac{2as}{(s^2+a^2)^2}$
$t\cos at$	$\leftrightarrow$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
$\sin at - at\cos at$	$\leftrightarrow$	$\dfrac{2a^3}{(s^2+a^2)^2}$
$\sin at + at\cos at$	$\leftrightarrow$	$\dfrac{2as^2}{(s^2+a^2)^2}$
$\sinh at$	$\leftrightarrow$	$\dfrac{a}{s^2-a^2}$
$\cosh at$	$\leftrightarrow$	$\dfrac{s}{s^2-a^2}$
$h(t-a)$	$\leftrightarrow$	$\dfrac{e^{-as}}{s}$
$\delta(t-a)$	$\leftrightarrow$	e^{-as}
$e^{at}f(t)$	$\leftrightarrow$	$F(s-a)$
$f(t)h(t-a)$	$\rightarrow$	$e^{-as}\mathcal{L}\{f(t+a)\}$
$f(t-a)h(t-a)$	$\leftarrow$	$e^{-as}F(s)$
$p-\text{periodic } f(t)$	$\rightarrow$	$\dfrac{1}{1-e^{-ps}}\displaystyle\int_0^p e^{-st}f(t)\,dt$
$\displaystyle\int_0^t f(u)g(t-u)\,du$	$\leftarrow$	$F(s)G(s)$
$t^nf(t) \quad (n=1,2,3,\ldots)$	$\leftrightarrow$	$(-1)^n\dfrac{d^nF}{ds^n}$
$f'(t)$	$\rightarrow$	$sF(s)-f(0)$
$f''(t)$	$\rightarrow$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$\rightarrow$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0)$

Table 6.2

19.



20.



In Exercises 21–32 use property 6.8a to find the Laplace transform for the function.

21. $f(t) = t^3 e^{-5t}$

22. $f(t) = t^2 e^{3t}$

23. $f(t) = 4te^{-t} - 2e^{-3t}$

24. $f(t) = 5e^{at} - 5e^{-at}$

25. $f(t) = e^t \sin 2t + e^{-t} \cos t$

26. $f(t) = 2e^{-3t} \sin 3t + 4e^{3t} \cos 3t$

27. $f(t) = te^t \cos 2t$

28. $f(t) = te^{-2t} \sin t$

29. $f(t) = 2e^t(\cos t + \sin t)$

30. $f(t) = (t-1)e^{2-3t} \sin 4t$

31. $f(t) = t^2 \cos at$

32. $f(t) = t^2 \sin at$

In Exercises 33–42 use property 6.14a to find the Laplace transform of the function.

33. $f(t) = (t-2)^2 h(t-2)$

34. $f(t) = \sin 3(t-4)\, h(t-4)$

35. $f(t) = t\, h(t-1)$

36. $f(t) = (t+5)\, h(t-3)$

37. $f(t) = (t^2+2)\, h(t-1)$

38. $f(t) = \cos t\, h(t-\pi)$

39. $f(t) = \cos t\, h(t-2)$

40. $f(t) = e^t h(t-4)$

41. $f(t) = t^2 e^t\, h(t-3)$

42. $f(t) = e^t \cos 2t\, h(t-1)$

In Exercises 43–47 find the Laplace transform of the periodic function.

43. $f(t) = t, \quad 0 < t < a, \quad f(t+a) = f(t)$

44. $f(t) = \begin{cases} 1, & 0 < t < a \\ -1, & a < t < 2a \end{cases} \qquad f(t+2a) = f(t)$

45. $f(t) = |\sin at|$

46. $f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases} \qquad f(t+2a) = f(t)$

47. $f(t) = \begin{cases} 1, & 0 < t < a \\ 0, & a < t < 2a \end{cases} \qquad f(t+2a) = f(t)$

Find the inverse Laplace transform in Exercises 48–69.

48. $F(s) = \dfrac{1}{s^2 - 2s + 5}$

49. $F(s) = \dfrac{s}{s^2 + 4s + 1}$

50. $F(s) = \dfrac{e^{-2s}}{s^2}$

51. $F(s) = \dfrac{e^{-3s}}{s^2 + 1}$

52. $F(s) = \dfrac{se^{-5s}}{s^2 + 2}$

53. $F(s) = \dfrac{se^{-s}}{(s^2 + 4)^2}$

54. $F(s) = \dfrac{1}{4s^2 - 6s - 5}$

55. $F(s) = \dfrac{s}{s^2 - 3s + 2}$

56. $F(s) = \dfrac{4s + 1}{(s^2 + s)(4s^2 - 1)}$

57. $F(s) = \dfrac{e^{-3s}}{s + 5}$

58. $F(s) = \dfrac{e^{-2s}}{s^2+3s+2}$

59. $F(s) = \dfrac{1}{s^3+1}$

60. $F(s) = \dfrac{5s-2}{3s^2+4s+8}$

61. $F(s) = \dfrac{e^{-s}(1-e^{-s})}{s(s^2+1)}$

62. $F(s) = \dfrac{s}{(s+1)^5}$

63. $F(s) = \dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

64. $F(s) = \dfrac{s^2}{(s^2-4)^2}$

65. $F(s) = \dfrac{1}{s(1-e^{-s})}$

66. $F(s) = \dfrac{1}{s(1+e^{-s})}$

67. $F(s) = \dfrac{1}{(s^2+4)(1-e^{-3s})}$

68. $F(s) = \dfrac{1}{(s^3+5s)(1-e^{-2s})}$

69. $F(s) = \dfrac{(s^2+1)e^{-2s}}{(s^4+2s^2)(1+e^{-s})}$

70. To find the inverse transform of a rational function with irreducible quadratic factors in denominators, we have used property 6.8b. Example 6.8 contained such a situation, and some of the above exercises. If you love to work with complex numbers, you might be pleased to know that you can always replace irreducible real factors with complex linear factors. It is not a method that we recommend, but it is at least comforting to know that it can be done. We illustrate with a simple example, hoping that it convinces you not to persue complex linear factors in the future.

(a) Use property property 6.8b to find the inverse transform of

$$F(s) = \frac{s+2}{s^2+2s+5}.$$

(b) Find the complex roots of $s^2+2s+5=0$, and use them to show that the partial fraction decomposition of $F(s)$ with complex linear factors is

$$F(s) = \frac{1}{4}\left(\frac{2-i}{s+1-2i} + \frac{2+i}{s+1+2i}\right).$$

(c) Use the decomposition in part (b) to find $\mathcal{L}^{-1}\{F(s)\}$.

71. The following two formulas, called **reduction of order formulas**, can be useful in taking inverse transforms,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^{n+1}}\right\} = \frac{t}{2n}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^n}\right\},$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^{n+1}}\right\} = \frac{-t}{2na^2}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^n}\right\} + \frac{2n-1}{2na^2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^n}\right\}.$$

Verify these formulas using

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}.$$

This result will be verified and extended in Section 6.7.

72. Use the reduction of order formulas in Exercise 71 to verify lines 5–8 in Table 6.2.

Use the reduction of order formulas in Exercise 71 to find the inverse Laplace transform in Exercises 73–76.

73. $F(s) = \dfrac{1}{(s^2+a^2)^3}$

74. $F(s) = \dfrac{s}{(s^2+a^2)^3}$

75. $F(s) = \dfrac{1}{(s^2 - 2s + 5)^3}$

76. $F(s) = \dfrac{s+2}{(s^2 - 4s + 13)^3}$

77. If $F(s) = \mathcal{L}\{f(t)\}$ for $s > \alpha$, for what values of s is $F(s-a)$ the Laplace transform of $e^{at}f(t)$?

78. Find the Laplace transform of the function

$$f(t) = \begin{cases} t^2/4, & 0 \le t < 1 \\ -(t^2 - 4t + 2)/4, & 1 \le t < 3 \\ (t-4)^2/4, & 3 \le t \le 4 \end{cases} \qquad f(t+4) = f(t).$$

79. Verify the *change of scale* property: If $F(s) = \mathcal{L}\{f(t\}$ for $s > \alpha$, then for $a > 0$,

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \qquad s > \alpha a.$$

6.3 Laplace Transforms and Differential Equations

The Laplace transform is a powerful technique for solving linear, ordinary and partial differential equations. It replaces differentiations with algebraic operations. Like the techniques of Chapter 4, the transform cannot be used on nonlinear problems. A simple example such as the following nonlinear equation illustrates why,

$$yy'' + 2y' + 3y = t^2.$$

You may have noticed that we have not developed a general formula for the Laplace transform of the product of two functions, and the reason is that there just isn't one. There are special cases such as when e^{at} multiplies another function, but not for a product such as yy''. This is why the transform is not applied to nonlinear problems.

The following theorem and its corollary simplify the process of applying the Laplace transform to linear differential equations.

Theorem 6.6 Suppose f is continuous for $t \geq 0$ with a piecewise-continuous first derivative on every finite interval $0 \leq t \leq T$. If f is $O(e^{\alpha t})$, then $\mathcal{L}\{f'\}$ exists for $s > \alpha$, and

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0). \tag{6.17}$$

(A more precise representation of the left side of this equation is $\mathcal{L}\{f'\}(s)$.)

Proof If t_j, $j = 1, \ldots, n$ denote the discontinuities of f' in $0 \leq t \leq T$, then

$$\int_0^T e^{-st} f'(t)\, dt = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} e^{-st} f'(t)\, dt,$$

where $t_0 = 0$ and $t_{n+1} = T$. Since f' is continuous on each subinterval, we may integrate by parts on these subintervals,

$$\int_0^T e^{-st} f'(t)\, dt = \sum_{j=0}^{n} \left[\left\{ e^{-st} f(t) \right\}_{t_j}^{t_{j+1}} + s \int_{t_j}^{t_{j+1}} e^{-st} f(t)\, dt \right].$$

Because f is continuous, $f(t_j+) = f(t_j-)$, $j = 1, \ldots, n$, and therefore

$$\int_0^T e^{-st} f'(t)\, dt = -f(0) + e^{-sT} f(T) + s \int_0^T e^{-st} f(t)\, dt.$$

Thus,

$$\begin{aligned}
\mathcal{L}\{f'\} &= \int_0^\infty e^{-st} f'(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-st} f'(t)\, dt \\
&= \lim_{T\to\infty} \left[-f(0) + e^{-sT} f(T) + s \int_0^T e^{-st} f(t)\, dt \right] \\
&= sF(s) - f(0) + \lim_{T\to\infty} e^{-sT} f(T),
\end{aligned}$$

provided the limit on the right exists. Since f is $O(e^{\alpha t})$, there exists M and $\overline{T}$ such that for $t > \overline{T}$, $|f(t)| < Me^{\alpha t}$. Thus, for $T > \overline{T}$,

$$e^{-sT} |f(T)| < e^{-sT} M e^{\alpha T} = M e^{(\alpha - s)T}$$

which approaches 0 as $T \to \infty$ (provided $s > \alpha$). Consequently,

$$\mathcal{L}\{f'\} = sF(s) - f(0). \blacksquare$$

This result is easily extended to second and higher order derivatives. For extensions when f is only piecewise-continuous, see Exercise 50.

Corollary 6.6.1 Suppose f and f' are continuous for $t \geq 0$, and f'' is piecewise-continuous on every finite interval $0 \leq t \leq T$. If f and f' are $O(e^{\alpha t})$, then $\mathcal{L}\{f''\}$ exists for $s > \alpha$, and

$$\mathcal{L}\{f''\} = s^2F(s) - sf(0) - f'(0). \tag{6.18}$$

Proof Since f' is continuous, f'' is piecewise-continuous, and f' is $O(e^{\alpha t})$, equation 6.17 gives

$$\mathcal{L}\{f''\} = s\mathcal{L}\{f'\} - f'(0).$$

We can apply equation 6.17 once again to obtain

$$\mathcal{L}\{f''\} = s[sF(s) - f(0)] - f'(0) = s^2F(s) - sf(0) - f'(0). \blacksquare$$

The extension to n^{th}-order derivatives is contained in the next corollary.

Corollary 6.6.2 Suppose f and its first $n-1$ derivatives are continuous for $t \geq 0$, and $f^{(n)}(t)$ is piecewise-continuous on every finite interval $0 \leq t \leq T$. If f and its first $n-1$ derivatives are $O(e^{\alpha t})$, then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > \alpha$, and

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{6.19}$$

In Section 6.1, we demonstrated how to use Laplace transforms to solve an initial-value problem. We now consider further examples.

Example 6.19 Solve the initial-value problem

$$y'' - 2y' + y = 2e^t, \quad y(0) = y'(0) = 0.$$

Solution First we assume that the solution of the problem is a function satisfying the conditions of Corollary 6.6.1. We can then take Laplace transforms of both sides of the differential equation,

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 2\mathcal{L}\{e^t\}.$$

Properties 6.17 and 6.18 yield

$$[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + Y(s) = \frac{2}{s-1}.$$

We now substitute from the initial conditions $y(0) = y'(0) = 0$,

$$s^2Y(s) - 2sY(s) + Y(s) = \frac{2}{s-1},$$

and solve this equation for $Y(s)$,

$$Y(s) = \frac{2}{(s-1)^3}.$$

The required function $y(t)$ can now be obtained by taking the inverse transform of $Y(s)$,

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} \qquad \text{(by linearity)}\\
&= 2e^t\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} \qquad \text{(by property 6.10b)}\\
&= 2e^t\left(\frac{t^2}{2}\right) \qquad \text{(from Table 6.1)}\\
&= t^2e^t.\bullet
\end{aligned}
$$

This example is typical of Laplace transforms at work on initial-value problems. We begin by assuming that the solution of the problem satisfies whatever conditions are necessary to apply the transform to the differential equation. In the case of Example 6.19, this meant assuming that $y(t)$ satisfies the conditions of Corollary 6.6.1. In actual fact we need only assume that $y(t)$ and $y'(t)$ are of exponential order. Since the nonhomogeneity $2e^t$ is continuous, our theory in Chapter 4 indicates that the solution has a continuous second derivative. In applying the Laplace transform to a third-order differential equation, we would assume that the solution satisfies the conditions of Corollary 6.6.2 for $n = 3$. The Laplace transform reduces the differential equation in $y(t)$ to an algebraic equation in its transform $Y(s)$. Notice how initial conditions for the solution of the initial-value problem are incorporated by the Laplace transform at a very early stage, unlike the techniques of Chapter 4 where they are used to determine arbitrary constants in a general solution. The algebraic equation is solved for $Y(s)$ and the inverse transform then yields a function $y(t)$. That $y(t)$ is a solution of the initial-value problem can be verified in two ways. First, we can check that $y(t)$ and $y'(t)$ are of exponential order, thus vindicating the initial assumption. Alternatively, we can verify that $y(t)$ satisfies the differential equation and initial conditions. We will omit these formal verifications, although the problem is not truly solved until one of these actions has been taken.

In each occurrence of the Laplace transform of $y(t)$ in the above example, we wrote $Y(s)$. In order to keep notation as simple as possible in further examples, we will write Y in place of $Y(s)$ when taking Laplace transforms of a differential equation.

Example 6.20 Solve the initial-value problem

$$y'' + 4y = 3\cos 2t, \qquad y(0) = 1, \quad y'(0) = 0.$$

Solution Assuming that the solution and its first derivative are of exponential order, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 0] + 4Y = \frac{3s}{s^2+4}.$$

The solution of this equation for $Y(s)$ is

$$Y(s) = \frac{3s}{(s^2+4)^2} + \frac{s}{s^2+4},$$

and Table 6.1 gives

$$y(t) = 3\left(\frac{t}{4}\sin 2t\right) + \cos 2t.\bullet$$

Laplace transforms thrive on initial-value problems; they use the initial conditions of the problem when the Laplace transform is applied to the derivative terms in the differential equation. They can also be adapted to boundary-value problems, as the following example illustrates.

Example 6.21 Solve the following boundary-value problem on the interval $0 \le t \le \pi/2$,

$$y'' + 9y = \cos 2t, \quad y(0) = 1, \quad y(\pi/2) = -1.$$

Solution The solution of the problem is only desired on the interval $0 \le t \le \pi/2$. What we do is solve the problem on the interval $t \ge 0$, and then restrict the solution to the interval $0 \le t \le \pi/2$. When we apply formula 6.18 to the second derivative in the differential equation, the derivative $y'(0)$ is needed. Since it is not one of the given pieces of information in the problem, we assign a letter to represent it; that is, we let $y'(0) = A$. If we assume that the solution and its first derivative are of exponential order and apply the Laplace transform to the differential equation, we obtain

$$[s^2 Y - s(1) - A] + 9Y = \frac{s}{s^2+4}.$$

We now solve for $Y(s)$,

$$Y(s) = \frac{s+A}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}.$$

Partial fractions on the second term gives

$$Y(s) = \frac{s+A}{s^2+9} + \frac{s/5}{s^2+4} + \frac{-s/5}{s^2+9} = \frac{4s/5+A}{s^2+9} + \frac{s/5}{s^2+4}.$$

Inverse transforms yield

$$y(t) = \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t + \frac{1}{5}\cos 2t.$$

The boundary condition $y(\pi/2) = -1$ can now be used to find A,

$$-1 = -\frac{A}{3} - \frac{1}{5} \quad \Longrightarrow \quad A = \frac{12}{5}.$$

The solution of the boundary-value problem is

$$y(t) = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t.\bullet$$

Laplace transforms are particularly adept at handling initial conditions, and as we have just seen, they can be adapted to boundary conditions. They can also provide general solutions to linear differential equations, as shown in the next example.

Example 6.22 Find a general solution of the differential equation $y'' + 2y' - 3y = t^2$.

Solution We denote initial values of the solution and its first derivative by $y(0) = A$ and $y'(0) = B$. If we assume that the solution and its first derivative are of exponential order, and take Laplace transforms of both sides of the differential equation,

$$[s^2Y - s(A) - B] + 2[sY - A] - 3Y = \frac{2}{s^3}.$$

The solution of this equation for $Y(s)$ is

$$Y(s) = \frac{2}{s^3(s^2+2s-3)} + \frac{As+(B+2A)}{s^2+2s-3}.$$

The partial fraction decomposition of the first term is

$$\frac{2}{s^3(s^2+2s-3)} = \frac{-2/3}{s^3} - \frac{4/9}{s^2} - \frac{14/27}{s} + \frac{1/2}{s-1} + \frac{1/54}{s+3}.$$

Hence,

$$Y(s) = \frac{-2/3}{s^3} - \frac{4/9}{s^2} - \frac{14/27}{s} + \frac{1/2}{s-1} + \frac{1/54}{s+3} + \frac{As+(B+2A)}{(s-1)(s+3)}.$$

If we are not concerned with preserving the fact that A and B represent initial values for $y(t)$ and its first derivative, we can write that $Y(s)$ is of the form

$$Y(s) = \frac{-2/3}{s^3} - \frac{4/9}{s^2} - \frac{14/27}{s} + \frac{C}{s-1} + \frac{D}{s+3},$$

where C and D are constants. Inverse transforms now give a general solution

$$y(t) = -\frac{t^2}{3} - \frac{4t}{9} - \frac{14}{27} + Ce^t + De^{-3t}. \bullet$$

Did you notice that the denominator of the transform in each of the above examples is the function $\phi(m)$ in the auxiliary equation of Chapter 4 with m replaced by s; that is, it is $\phi(s)$. This is always the case for linear differential equations with constant coefficients; and it can serve as a partial check on calculations.

Example 6.23 A 2-kilogram mass is suspended from a spring with constant 128 newtons per metre. It is pulled 4 centimetres above its equilibrium position and released. An external force $3\sin\omega t$ newtons acts vertically on the mass during its motion. If damping is negligible, find the position of the mass as a function of time.

Solution The initial-value problem describing oscillations of the mass is

$$2\frac{d^2x}{dt^2} + 128x = 3\sin\omega t, \quad x(0) = 1/25, \quad x'(0) = 0.$$

If we take Laplace transforms of both sides of the differential equation,

$$2[s^2X - s/25] + 128X = \frac{3\omega}{s^2+\omega^2} \implies X(s) = \frac{3\omega}{2(s^2+64)(s^2+\omega^2)} + \frac{s}{25(s^2+64)}.$$

When $\omega \neq 8$, partial fractions on the first term on the left leads to

$$X(s) = \frac{3\omega}{2(64-\omega^2)(s^2+\omega^2)} - \frac{3\omega}{2(64-\omega^2)(s^2+64)} + \frac{s}{25(s^2+64)}.$$

Hence, displacement in the absence of resonance is

$$x(t) = \frac{3}{2(64-\omega^2)}\sin\omega t - \frac{3\omega}{16(64-\omega^2)}\sin 8t + \frac{1}{25}\cos 8t.$$

When $\omega = 8$, the Laplace transform $X(s)$ takes the form

$$X(s) = \frac{12}{(s^2+64)^2} + \frac{s}{25(s^2+64)},$$

in which case Table 6.1 gives the resonant solution

$$x(t) = \frac{12}{2(8)^3}(\sin 8t - 8t\,\cos 8t) + \frac{1}{25}\cos 8t = \frac{3}{256}\sin 8t - \frac{3t}{32}\cos 8t + \frac{1}{25}\cos 8t.\bullet$$

Convolutions

As a linear operator, the Laplace transform efficiently handles sums and differences of functions. What it does not handle is the product of functions; that is, we do not have a formula for the Laplace transform of the product of two functions $f(t)$ and $g(t)$. We can take the transform of certain products such as an exponential multiplying another function (formula 6.8). In Section 6.7, we also find out how to take the Laplace transform of the product $t^n f(t)$, where n is a positive integer. But, in general, there is no formula for the Laplace transform of the product of two arbitrary functions. One would expect that there would therefore be no formula for the inverse Laplace transform of the product of two functions. Surprisingly, there is a formula for $\mathcal{L}^{-1}\{F(s)G(s)\}$ when inverse transforms of $F(s)$ and $G(s)$ are known. We shall see shortly that the inverse of $F(s)G(s)$ is what is called the *convolution* of $f(t)$ and $g(t)$.

Definition 6.4 The **convolution** of two functions f and g is a function denoted by $f*g$ with values defined by

$$(f*g)(t) = \int_0^t f(u)g(t-u)\,du. \tag{6.20}$$

The following properties of convolutions are easily verified using Definition 6.4:

$$f*g = g*f, \tag{6.21a}$$

$$f*(kg) = (kf)*g = k(f*g),\ \ k \text{ a constant} \tag{6.21b}$$

$$(f*g)*h = f*(g*h), \tag{6.21c}$$

$$f*(g+h) = f*g + f*h. \tag{6.21d}$$

Example 6.24 Find the convolution of $f(t) = \sin t$ and $g(t) = \cos 4t$.

Solution According to equation 6.20,

$$(f*g)(t) = \int_0^t \sin u\,\cos 4(t-u)\,du.$$

With the trigonometric identity $\sin A\cos B = (1/2)[\sin(A+B) + \sin(A-B)]$, we obtain

$$\begin{aligned}(f*g)(t) &= \frac{1}{2}\int_0^t [\sin(4t-3u) + \sin(5u-4t)]\,du \\ &= \frac{1}{2}\left\{\frac{1}{3}\cos(4t-3u) - \frac{1}{5}\cos(5u-4t)\right\}_0^t \\ &= \frac{1}{15}(\cos t - \cos 4t).\bullet\end{aligned}$$

The importance of convolutions lies in the following theorem.

Theorem 6.7 If f and g are $O(e^{\alpha t})$ and piecewise-continuous on every finite interval $0 \le t \le T$, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}, \quad s > \alpha. \tag{6.22a}$$

Proof If $F = \mathcal{L}\{f\}$ and $G = \mathcal{L}\{g\}$, then

$$F(s)G(s) = \int_0^\infty e^{-su} f(u)\, du \int_0^\infty e^{-s\tau} g(\tau)\, d\tau = \int_0^\infty \int_0^\infty e^{-s(u+\tau)} f(u)g(\tau)\, d\tau\, du.$$

Suppose we change variables of integration in the inner integral with respect to τ by setting $t = u + \tau$. Then

$$F(s)G(s) = \int_0^\infty \int_u^\infty e^{-st} f(u)g(t-u)\, dt\, du = \lim_{T\to\infty} \int_0^T \int_u^\infty e^{-st} f(u)g(t-u)\, dt\, du.$$

We would like to interchange orders of integration, but to do so requires that the inner integral converge uniformly with respect to u. To verify that this is indeed the case we note that since f and g are $O(e^{\alpha t})$ and piecewise-continuous on every finite interval $0 \le t \le T$, there exists a constant M such that for all $t \ge 0$, $|f(t)| < Me^{\alpha t}$ and $|g(t)| < Me^{\alpha t}$. For each $u \ge 0$, we therefore have $|e^{-st} f(u)g(t-u)| < M^2 e^{-st} e^{\alpha u} e^{\alpha(t-u)} = M^2 e^{-t(s-\alpha)}$. Thus,

$$\left| \int_u^\infty e^{-st} f(u)g(t-u)\, dt \right| < M^2 \int_u^\infty e^{-t(s-\alpha)}\, dt = M^2 \left\{ \frac{e^{-t(s-\alpha)}}{\alpha - s} \right\}_u^\infty$$
$$= \frac{M^2 e^{-u(s-\alpha)}}{s-\alpha} < \frac{M^2}{s-\alpha},$$

provided $s > \alpha$, and the improper integral is uniformly convergent with respect to u. The order of integration in the expression for $F(s)G(s)$ may therefore be interchanged (Figure 6.26), and we obtain

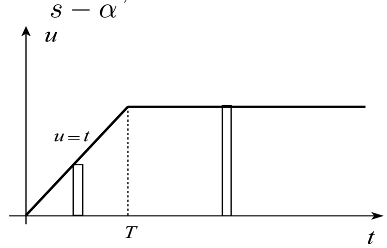


Figure 6.26

$$F(s)G(s) = \lim_{T\to\infty} \left[\int_0^T e^{-st} \int_0^t f(u)g(t-u)\, du\, dt + \int_T^\infty e^{-st} \int_0^T f(u)g(t-u)\, du\, dt \right].$$

Since

$$\left| \int_T^\infty e^{-st} \int_0^T f(u)g(t-u)\, du\, dt \right| < \int_T^\infty \int_0^T M^2 e^{-t(s-\alpha)} du\, dt$$
$$= M^2 T \left\{ \frac{e^{-t(s-\alpha)}}{\alpha - s} \right\}_T^\infty = \frac{M^2 T e^{-T(s-\alpha)}}{s-\alpha}$$

provided $s > \alpha$, it follows that

$$\lim_{T\to\infty} \int_T^\infty e^{-st} \int_0^T f(u)g(t-u)\,du\,dt = 0.$$

Thus,

$$F(s)G(s) = \lim_{T\to\infty} \int_0^T e^{-st} \int_0^t f(u)g(t-u)\,du\,dt = \lim_{T\to\infty} \int_0^T e^{-st} f * g\,dt = \mathcal{L}\{f * g\}. \ \blacksquare$$

More important in practice is the inverse of property 6.22a.

Corollary 6.7.1 If $\mathcal{L}^{-1}\{F\} = f$ and $\mathcal{L}^{-1}\{G\} = g$, where f and g are $O(e^{\alpha t})$ and piecewise-continuous on every finite interval, then

$$\mathcal{L}^{-1}\{FG\} = f * g. \tag{6.22b}$$

This is line 17 in Table 6.2. The following example illustrates how to use this corollary.

Example 6.25 Find the inverse transform of $F(s) = \dfrac{2}{s^2(s^2+4)}$.

Solution Since $\mathcal{L}^{-1}\{2/(s^2+4)\} = \sin 2t$ and $\mathcal{L}^{-1}\{1/s^2\} = t$, convolution property 6.22b gives

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{s^2(s^2+4)}\right\} &= \int_0^t u \sin 2(t-u)\,du \\ &= \left\{\frac{u}{2}\cos 2(t-u) + \frac{1}{4}\sin 2(t-u)\right\}_0^t = \frac{t}{2} - \frac{1}{4}\sin 2t.\end{aligned}$$

The alternative is to use partial fractions.•

Convolutions are particularly useful when solving differential equations that contain unspecified nonhomogeneities.

Example 6.26 Find the solution of the initial-value problem

$$y'' + 2y' + 3y = f(t), \qquad y(0) = 1, \quad y'(0) = 0,$$

where $f(t)$ is of exponential order and piecewise-continuous for $t \geq 0$.

Solution The only technique from Chapter 4 that can handle this problem is variation of parameters; the other techniques require that the form of the nonhomogeneity be known. To show that Laplace transforms can also be used, we assume that the solution and its first derivative are of exponential order, and take Laplace transforms of both sides of the differential equation,

$$[s^2Y - s] + 2[sY - 1] + 3Y = F(s).$$

We solve for $Y(s)$,

$$Y(s) = \frac{F(s)}{s^2+2s+3} + \frac{s+2}{s^2+2s+3}.$$

To find the inverse transform of this function, we first note that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} = \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t.$$

Convolution property 6.22b on the first term of $Y(s)$ now yields

$$\begin{aligned} y(t) &= \int_0^t f(u)\frac{1}{\sqrt{2}}e^{-(t-u)}\sin\sqrt{2}(t-u)\,du + \mathcal{L}^{-1}\left\{\frac{(s+1)+1}{(s+1)^2+2}\right\} \\ &= \frac{1}{\sqrt{2}}\int_0^t f(u)e^{-(t-u)}\sin\sqrt{2}(t-u)\,du + e^{-t}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\} \\ &= \frac{1}{\sqrt{2}}\int_0^t f(u)e^{-(t-u)}\sin\sqrt{2}(t-u)\,du + e^{-t}\left(\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right). \bullet \end{aligned}$$

The following nontrivial problem makes clever use of convolutions.

The Tautochrone

A bead, with zero initial velocity is to slide frictionlessly down a wire from a point $P(x,y)$ to the origin (Figure 6.27). Tautochrone is the name attached to the shape of the curve for which the time of descent is independent of the height, y, on the curve from which the bead starts. Suppose we let the equation of the curve be $x = x(y)$. Since the kinetic energy of the bead at any point (η,ζ) along the curve is equal to the initial potential energy of the bead, we can write that

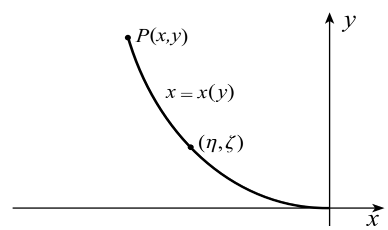


Figure 6.27

$$\frac{1}{2}mv^2 = mg(y-\zeta),$$

where m is the mass of the bead and v is its velocity. It follows that the velocity of the bead at (η,ζ) is

$$v = \sqrt{2g}\sqrt{y-\zeta}.$$

Since the time for the bead to traverse an element of arc length $\sqrt{d\eta^2+d\zeta^2}$ at point (η,ζ) is this length divided by v, the total time to travel from P to the origin is

$$T = \int_0^y \frac{\sqrt{d\eta^2+d\zeta^2}}{\sqrt{2g}\sqrt{y-\zeta}} = \frac{1}{\sqrt{2g}}\int_0^y \frac{\sqrt{1+\left(\frac{d\eta}{d\zeta}\right)^2}}{\sqrt{y-\zeta}}d\zeta.$$

If we set $f(\zeta) = \sqrt{1+(d\eta/d\zeta)^2}$, then

$$T = \frac{1}{\sqrt{2g}}\int_0^y \frac{f(\zeta)}{\sqrt{y-\zeta}}d\zeta.$$

This integral can be interpreted as the convolution of the functions $f(y)$ and $1/\sqrt{y}$. If we take Laplace transforms of both sides of the equation with respect to y, and note that T is a constant, we obtain

$$\frac{\sqrt{2g}T}{s} = \mathcal{L}\left\{f(y) * \frac{1}{\sqrt{y}}\right\}.$$

According to Exercise 35 in Section 6.1, the Laplace transform of $1/\sqrt{y}$ is $\sqrt{\pi/s}$. Thus,

$$\frac{\sqrt{2g}T}{s} = F(s)\sqrt{\frac{\pi}{s}} \qquad \Longrightarrow \qquad F(s) = \frac{\sqrt{2g}T}{\pi}\sqrt{\frac{\pi}{s}}.$$

The inverse transform now gives

$$f(y) = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{\sqrt{2g}T}{\pi}\frac{1}{\sqrt{y}},$$

a differential equation for $x(y)$. It can be rewritten in the form

$$\frac{dx}{dy} = -\sqrt{\frac{k^2 - y}{y}},$$

where we have set $k = \sqrt{2g}T/\pi$. We now integrate and set $y = k^2 \sin^2\theta$,

$$\begin{aligned}
-x + C &= \int \frac{\sqrt{k^2 - y}}{\sqrt{y}}dy = \int \frac{k\cos\theta}{k\sin\theta}2k^2\sin\theta\cos\theta\,d\theta \\
&= 2k^2\int \cos^2\theta\,d\theta = 2k^2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = k^2\left(\theta + \frac{1}{2}\sin 2\theta\right) \\
&= k^2\left[\mathrm{Sin}^{-1}\left(\frac{\sqrt{y}}{k}\right) + \frac{\sqrt{y}}{k}\sqrt{1 - \frac{y}{k^2}}\right] \\
&= k^2\mathrm{Sin}^{-1}\left(\frac{\sqrt{y}}{k}\right) + \sqrt{k^2 y - y^2}.
\end{aligned}$$

To pass through the origin, C must be zero, so that the equation of the tautochrone is

$$x = -k^2\mathrm{Sin}^{-1}\left(\frac{\sqrt{y}}{k}\right) - \sqrt{k^2 y - y^2}.$$

By setting $\phi = -2\theta$, parametric equations for the tautochrone are

$$\begin{aligned}
x &= -\frac{k^2}{2}(2\theta + \sin 2\theta) = \frac{k^2}{2}(\phi + \sin\phi), \\
y &= k^2\sin^2\theta = k^2\left(\frac{1 - \cos 2\theta}{2}\right) = \frac{k^2}{2}(1 - \cos\phi).
\end{aligned}$$

These are parametric equations for a cycloid (see Exercise 60 in Section 2.2).

EXERCISES 6.3

In Exercises 1–16 use Laplace transforms to solve the initial-value problem.

1. $y'' + 3y' - 4y = t + 3, \quad y(0) = 1, \quad y'(0) = 0$

2. $y'' + 2y' - y = e^t, \quad y(0) = 1, \quad y'(0) = 2$

3. $y'' + y = 2e^{-t}, \quad y(0) = y'(0) = 0$

4. $y'' + 2y' + y = t, \quad y(0) = 0, \quad y'(0) = 1$

5. $y'' - 2y' + y = t^2 e^t, \quad y(0) = 1, \quad y'(0) = 0$

6. $y'' + y = t, \quad y(0) = 1, \quad y'(0) = -2$

7. $y'' + 2y' + 5y = e^{-t} \sin t, \quad y(0) = 0, \quad y'(0) = 1$

8. $y'' + 6y' + y = \sin 3t, \quad y(0) = 2, \quad y'(0) = 1$

9. $y'' + y' - 6y = t + \cos t, \quad y(0) = 1, \quad y'(0) = -2$

10. $y'' - 4y' + 5y = te^{-3t}, \quad y(0) = -1, \quad y'(0) = 2$

11. $y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 1$, where $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

12. $y'' + 2y' - 4y = \cos^2 t, \quad y(0) = 0, \quad y'(0) = 0$

13. $y'' - 3y' + 2y = 8t^2 + 12e^{-t}, \quad y(0) = 0, \quad y'(0) = 2$

14. $y'' + 4y' - 2y = \sin 4t, \quad y(0) = 0, \quad y'(0) = 0$

15. $y'' + 8y' + 41y = e^{-2t} \sin t, \quad y(0) = 0, \quad y'(0) = 1$

16. $y'' + 2y' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$, where $f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

In Exercises 17–19 use Laplace transforms to solve the boundary-value problem.

17. $y'' + 9y = \cos 2t, \quad y(0) = 1, \quad y(\pi/2) = -1$

18. $y'' + 3y' - 4y = 2e^{-4t}, \quad y(0) = 1, \quad y(1) = 1$

19. $y'' + 2y' + 5y = e^{-t} \sin t, \quad y(0) = 0, \quad y(\pi/4) = 1$

20. There are two ways to use Laplace transforms to solve the initial-value problem

$$y'' + 2y' - 3y = \sin 2t, \qquad y(\pi/2) = 3, \quad y'(\pi/2) = -1.$$

Do both:
(a) Find a general solution and then use the initial conditions to find constants.
(b) Translate the initial conditions to $u = 0$ by setting $u = t - \pi/2$. Solve the problem for $y(u)$, and then return to $y(t)$.

In Exercises 21–24 use Laplace transforms to find an integral representation for the solution to the problem.

21. $y'' - 4y' + 3y = f(t), \quad y(0) = 1, \quad y'(0) = 0$

22. $y'' + 4y' + 6y = f(t), \quad y(0) = 0, \quad y'(0) = 0$

23. $y'' + 16y = f(t)$

24. $y'' + 3y' + 2y = e^t f(t)$

In Exercises 25–28 use convolutions to find the inverse Laplace transform for the function.

25. $F(s) = \dfrac{1}{s(s+1)}$

26. $F(s) = \dfrac{1}{(s^2+1)(s^2+4)}$

27. $F(s) = \dfrac{s}{(s+4)(s^2-2)}$

28. $F(s) = \dfrac{s}{(s^2-4)(s^2-9)}$

In Exercises 29–34 use Laplace transforms to find a general solution of the differential equation.

29. $y'' - 2y' + 4y = t^2$

30. $y'' - 2y' + y = t^2e^t$

31. $y'' + y = f(t)$

32. $y'' + 2y' + 5y = e^{-t}\sin t$

33. $y'' + 4y' + y = t + 2$

34. $y'' - 4y = f(t)$

35. To find a general solution for $y'' + 9y = t\,\sin t$, replace $t\,\sin t$ by te^{ti}, solve the equation, and then take imaginary parts.

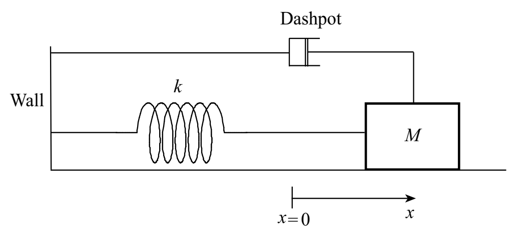
36. To find a general solution for $y'' - 2y' + 3y = t\,\cos 2t$, replace $t\,\cos 2t$ by te^{2ti}, solve the equation, and then take real parts.

Solve the problem in Exercises 37–38.

37. $y''' - 3y'' + 3y' - y = t^2e^t, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2$

38. $y''' - 3y'' + 3y' - y = t^2e^t$

One end of a spring with constant k newtons per metre is attached to a mass of M kilograms and the other end is attached to a wall (figure below).



Attached to the mass is a dashpot that provides, or represents, a resistive force on the mass directly proportional to the velocity of the mass. If all other forces are grouped into a function denoted by $f(t)$, the differential equation governing motion of the mass is

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t),$$

where $\beta > 0$ is a constant. The position of M when the spring is unstretched corresponds to $x = 0$. Accompanying the differential equation will be two initial conditions $x(0) = x_0$ and $x'(0) = v_0$ representing the initial position and velocity of M. In Exercises 39–45, solve the initial-value problem with the given information.

39. $M = 1/5, \quad \beta = 0, \quad k = 10, \quad f(t) = 0, \quad x(0) = -0.03, \quad x'(0) = 0$

40. $M = 2, \quad \beta = 0, \quad k = 16, \quad f(t) = 0, \quad x(0) = 0.1, \quad x'(0) = 0$

41. $M = 1/5, \quad \beta = 3/2, \quad k = 10, \quad f(t) = 0, \quad x(0) = -0.03, \quad x'(0) = 0$

42. $M = 1/5, \quad \beta = 3/2, \quad k = 10, \quad f(t) = 4\sin 10t, \quad x(0) = 0, \quad x'(0) = 0$

43. $M = 1/10, \quad \beta = 1/20, \quad k = 5, \quad f(t) = 0, \quad x(0) = -1/20, \quad x'(0) = 2$

44. $M = 1/10, \quad \beta = 0, \quad k = 4000, \quad f(t) = 3\cos 200t, \quad x(0) = 0, \quad x'(0) = 10$

45. $M = 1, \quad \beta = 0, \quad k = 64, \quad f(t) = 2\sin 8t, \quad x(0) = 0, \quad x'(0) = 0$

If the first or second derivative of a function $f(t)$ yields functions with known Laplace transforms and/or $f(t)$, then equations 6.17 and 6.18 can be used to find the Laplace transform of $f(t)$. We illustrate this in Exercises 46–49.

46. Find the Laplace transform of $\sin^2 at$ by:
(a) using a trigonometric identity;
(b) equation 6.17 and the fact that $\mathcal{L}\{\sin at\} = a/(s^2+a^2)$.

47. Find the Laplace transform of $t\cos at$ by calculating its second derivative and using equation 6.18.

48. In Example 6.5, we used multiple integrations by parts to find the Laplace transform of t^n when n is a nonnegative integer. Use equation 6.17 and mathematical induction to verify the transform.

49. (a) Use equation 6.17 and Exercise 35 in Section 6.1 to find the Laplace transform of $\sqrt{t}$.
(b) Extend the result in part (a) to find the Laplace transform of $t^{(2n+1)/2}$ when $n \geq 0$ is an integer.

50. (a) Let f be $O(e^{\alpha t})$ and be continuous for $t \geq 0$ except for a finite discontinuity at $t = t_0 > 0$; and let f' be piecewise continuous on every finite interval $0 \leq t \leq T$. Show that

$$\mathcal{L}\{f'\} = sF(s) - f(0) - e^{-st_0}[f(t_0+) - f(t_0-)].$$

(b) What is the result in part (a) if $t_0 = 0$?

51. (a) Calculate the convolution of t^m and t^n when m and n are positive integers.
(b) Calculate the convolution of t^m and t^n when $m > 0$ and $n > 0$ are not positive integers.

52. Verify that when $f(t)$ satisfies the conditions of Theorem 6.6, then its Laplace transform satisfies the equation

$$\lim_{s\to\infty} [sF(s)] = f(0).$$

This is called the **initial-value theorem**. Like Theorem 6.3, it can serve as a partial check on taking the Laplace transform of a given function $f(t)$.

53. Suppose that $f(t)$ satisfies the conditions of Theorem 6.6 and that $f'(t)$ is also of exponential order. Verify that if $\lim_{t\to\infty} f(t)$ exists, then

$$\lim_{s\to 0} s\,F(s) = \lim_{t\to\infty} f(t).$$

This is called the **final-value theorem**.

54. Prove that the derivative of the function $\sin(e^{t^2})$ in Exercise 45 of Section 6.1 has a Laplace transform.

6.4 Piecewise-defined and Discontinuous Nonhomogeneities

Nonhomogeneities for the linear differential equations in Section 6.3 were all continuous. As a result, Laplace transforms did not prove overly advantageous compared to methods of Chapter 4. In this section we show that Laplace transforms are exceptional for handling discontinuous and piecewise-defined nonhomogeneities. But, before doing so, we examine the expectations of solutions to linear differential equations that contain piecewise-continuous nonhomogeneities.

We begin with the initial-value problem associated with a linear first-order differential equation,

$$\frac{dy}{dt} + P(t)y = Q(t), \quad y(0) = y_0, \tag{6.23}$$

on the interval $t > 0$, where $P(t)$ is continuous. When $Q(t)$ is also continuous, the solution $y(t)$ of the initial-value problem is unique. It is continuous and has a continuous first derivative. But what can we say about the solution if $Q(t)$ is piecewise-continuous? Suppose, for example, that $Q(t)$ has a single, finite-jump discontinuity at some value $t_0 > 0$. The initial-value problem has a continuous solution with continuous derivative on the interval $0 < t < t_0$, call it $y_1(t)$. The differential equation also has a general solution, call it $y_2(t)$ on the interval $t > t_0$. We can match these solutions at t_0 by demanding that the solution be continuous at t_0. This requires

$$\lim_{t\to t_0^-} y_1(t) = \lim_{t\to t_0^+} y_2(t).$$

This determines the arbitrary constant in the general solution $y_2(t)$. It creates a continuous function that satisfies the differential equation at every value of t except t_0. The differential equation requires $dy/dt = Q(t) - P(t)y(t)$. Because $y(t)$ and $P(t)$ are continuous, but $Q(t)$ is discontinuous at t_0, it follows that $y(t)$ is differentiable at every value of t except t_0. Since we can match solutions at every discontinuity of $Q(t)$, we have the following theorem.

Theorem 6.8 When $Q(t)$ is piecewise-continuous on the interval $t > 0$, there exists a unique solution of initial-value problem 6.23 that is continuous and has a continuous first derivative except at points of discontinuity of $Q(t)$. It therefore satisfies the differential equation except at discontinuities of $Q(t)$.

Here is an example to illustrate this matching.

Example 6.27 Find the solution of the following initial-value problem with a piecewise-continuous nonhomogeneity

$$\frac{dy}{dt} + 3y = f(t), \qquad \text{where } f(t) = \begin{cases} t, & 0 < t < 1 \\ 2, & t > 1, \end{cases}$$

subject to $y(0) = 1$.

Solution First we solve the differential equation on the interval $0 < t < 1$, in which case it is

$$\frac{dy}{dt} + 3y = t.$$

An integrating factor is e^{3t}, so that multiplication of the differential equation by e^{3t} results in

$$e^{3t}\frac{dy}{dt} + 3ye^{3t} = te^{3t} \qquad \text{or} \qquad \frac{d}{dt}(ye^{3t}) = te^{3t}.$$

Antidifferentiation gives

$$ye^{3t} = \int te^{3t}\,dt = \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} + C,$$

and therefore

$$y(t) = \frac{t}{3} - \frac{1}{9} + Ce^{-3t}.$$

The initial condition $y(0) = 1$ requires $1 = -1/9 + C$, and therefore $C = 10/9$. The solution on the interval $0 < t < 1$ is

$$y(t) = \frac{t}{3} - \frac{1}{9} + \frac{10}{9}e^{-3t}.$$

We now consider the differential equation on the interval $t > 1$,

$$\frac{dy}{dt} + 3y = 2.$$

Once again e^{3t} is an integrating factor, and this leads to the solution

$$y(t) = \frac{2}{3} + De^{-3t}, \quad \text{for } t > 1.$$

What remains is to evaluate constant D. According to Theorem 6.8, there is a solution that is continuous for all t, and in particular at $t = 1$. This requires

$$\lim_{t\to 1^-} y(t) = \lim_{y\to 1^+} y(t) \quad \Longrightarrow \quad \lim_{t\to 1^-}\left[\frac{t}{3} - \frac{1}{9} + \frac{10}{9}e^{-3t}\right] = \lim_{t\to 1^+}\left[\frac{2}{3} + De^{-3t}\right].$$

Evaluating the limits gives

$$\frac{1}{3} - \frac{1}{9} + \frac{10}{9}e^{-3} = \frac{2}{3} + De^{-3}, \quad \text{from which} \quad D = \frac{1}{9}(10 - 4e^3).$$

Thus, the solution of the initial-value problem is the function

$$y(t) = \begin{cases} \dfrac{t}{3} - \dfrac{1}{9} + \dfrac{10}{9}e^{-3t}, & 0 \le t \le 1 \\ \dfrac{2}{3} + \dfrac{1}{9}(10 - 4e^3)e^{-3t}, & t > 1. \bullet \end{cases}$$

It is graphed in Figure 6.28. As predicted by Theorem 6.8, it is continuous, even at the discontinuity $t = 1$ of $f(t)$, but it does not have a derivative there.$\bullet$

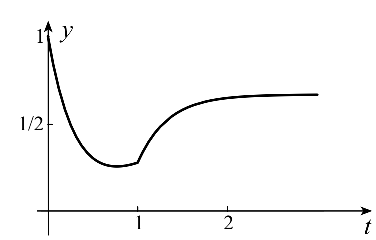


Figure 6.28

We now give a similar discussion for initial-value problems associated with second-order, linear differential equations,

$$a_2(t)\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t), \quad y(0) = y_0, \quad y'(0) = y_0'. \tag{6.24}$$

We assume as usual that $a_2(t)$, $a_1(t)$, and $a_0(t)$ are continuous for $t \geq 0$, and $a_2(t) \neq 0$ for any value of $t \geq 0$. When $f(t)$ is also continuous, the solution $y(t)$ of the initial-value problem is unique. It is continuous and has continuous first and second derivatives. Suppose, however, that $f(t)$ has a single, finite-jump discontinuity at some value $t_0 > 0$. The initial-value problem has a continuous solution with continuous first and second derivatives on the interval $0 < t < t_0$, call it $y_1(t)$. The differential equation also has a general solution, call it $y_2(t)$ on the interval $t > t_0$. We can match these solutions at t_0 by demanding that

$$\lim_{t \to t_0^-} y_1(t) = \lim_{t \to t_0^+} y_2(t), \qquad \lim_{t \to t_0^-} y_1'(t) = \lim_{t \to t_0^+} y_2'(t).$$

This determines the arbitrary constants in the general solution $y_2(t)$. It creates a continuous function with a continuous first derivative that satisfies the differential equation at every value of t except t_0. Because $d^2y/dt^2 = f(t)/a_2(t) - [a_1(t)/a_2(t)]dy/dt - [a_0(t)/a_2(t)]y$, $y(t)$ has a second derivative at every value of t except t_0. Since we can match solutions at every discontinuity of $Q(t)$, we have the following theorem.

Theorem 6.9 When $f(t)$ is piecewise-continuous on the interval $t > 0$, there exists a unique solution of initial-problem 6.24 that is continuous, with a continous first derivative, and has a continuous second derivative except at points of discontinuity of $f(t)$. It therefore satisfies the differential equation except at discontinuities of $f(t)$.

The following example illustrates this matching.

Example 6.28 Solve the following initial-value problem with a piecewise-continuous nonhomogeneity,

$$y'' + 2y' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0,$$

where $f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1. \end{cases}$

Solution The auxiliary equation $m^2 + 2m + 1 = 0$ has double root $m = -1$. On the interval $0 < t < 1$, a particular solution of the differential equation is $y_p = t - 2$, and hence a general solution on this interval is $y_1(t) = (C_1 + C_2t)e^{-t} + t - 2$. The initial conditions require

$$1 = y(0) = C_1 - 2, \qquad 0 = y'(0) = C_2 - C_1 + 1,$$

the solution of which is $C_1 = 3$ and $C_2 = 2$. On the interval $0 < t < 1$, then,

$$y_1(t) = (3 + 2t)e^{-t} + t - 2.$$

For $t > 1$, a general solution of the differential equation is $y_2(t) = (D_1 + D_2t)e^{-t}$.

For the solution to be continuous and have a continuous first derivative at $t = 1$, we must have

$$\lim_{t \to 1^-} y_1(t) = \lim_{t \to 1^+} y_2(t), \qquad \lim_{t \to 1^-} y_1'(t) = \lim_{t \to 1^+} y_2'(t).$$

Substitution for $y_1(t)$ and $y_2(t)$ gives

$$5e^{-1}-1=(D_1+D_2)e^{-1}, \qquad -3e^{-1}+1=-D_1e^{-1}.$$

These can be solved for $D_1 = 3 - e$ and $D_2 = 2$, and therefore the solution of the initial-value problem is

$$y(t)=\begin{cases}(3+2t)e^{-t}+t-2, & 0\le t\le 1\\ (3-e+2t)e^{-t}, & t>1.\bullet\end{cases}$$

It is graphed in Figure 6.29. As predicted by Theorem 6.9, it is continuous, and appears to have a continuous first derivative at the discontinuity $t = 1$ of $f(t)$. The second derivative is discontinuous at $t = 1$, but we cannot see this graphically.●

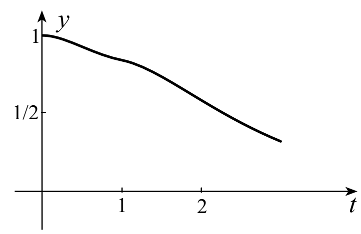


Figure 6.29

This procedure of matching solutions at finite discontinuities of nonhomogeneities can be extended to include initial-value problems associated with n^{th}-order, linear differential equations

$$a_n(t)\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_1(t)\frac{dy}{dt}+a_0(t)y=f(t), \tag{6.25a}$$

subject to initial conditions

$$y_1(0)=y_0, \quad y'(0)=y_0', \quad \cdots \quad ,y^{(n-1)}(0)=y_0^{(n-1)}. \tag{6.25b}$$

At each discontinuity of $f(t)$, the function and its first $n-1$ derivatives are matched to produce a solution that has continuous derivatives of orders up to and including $n - 1$, but a discontinuity in the n^{th} derivative results.

What is most important to realize from the above discussion is that matching a solution and its derivatives at discontinuities of a piecewise-continuous nonhomogeneity is tedious, especially as the number of discontinuities increases. Laplace transforms provide an excellent alternative. If we apply Laplace transforms to the initial-value problem of Example 6.27, we get

$$sY-1+3Y=\mathcal{L}\{f(t)\},$$

where

$$\begin{aligned}\mathcal{L}\{f(t)\}&=\mathcal{L}\{t[h(t)-h(t-1)]+2\,h(t-1)\}=\mathcal{L}\{t+(2-t)h(t-1)\}\\&=\frac{1}{s^2}+e^{-s}\mathcal{L}\{2-(t+1)\}=\frac{1}{s^2}+e^{-s}\left(\frac{1}{s}-\frac{1}{s^2}\right).\end{aligned}$$

Thus,

$$Y(s)=\frac{1}{s+3}\left[1+\frac{1}{s^2}+e^{-s}\left(\frac{1}{s}-\frac{1}{s^2}\right)\right]=\frac{1+s^2}{s^2(s+3)}+\frac{e^{-s}(s-1)}{s^2(s+3)}.$$

Partial fractions and inverse transforms give

$$\begin{aligned}y(t)&=\mathcal{L}^{-1}\left\{\frac{-1/9}{s}+\frac{1/3}{s^2}+\frac{10/9}{s+3}+e^{-s}\left(\frac{4/9}{s}-\frac{1/3}{s^2}-\frac{4/9}{s+3}\right)\right\}\\&=-\frac{1}{9}+\frac{t}{3}+\frac{10}{9}e^{-3t}+\left[\frac{4}{9}-\frac{1}{3}(t-1)-\frac{4}{9}e^{-3(t-1)}\right]h(t-1).\end{aligned}$$

This is the solution obtained in Example 6.27.

We now solve the initial-value problem in Example 6.28 using Laplace transforms. When we take transforms of the differential equation, we get

$$[s^2Y - s] + 2[sY - 1] + Y = \mathcal{L}\{f(t)\},$$

where

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{t\,[h(t) - h(t-1)]\} = \mathcal{L}\{t - \mathcal{L}\{t\,h(t-1)\} \\ &= \frac{1}{s^2} - e^{-s}\mathcal{L}\{t+1\} = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right).\end{aligned}$$

Thus,

$$\begin{aligned}Y(s) &= \frac{1}{(s+1)^2}\left[s + 2 + \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)\right] \\ &= \frac{(s+1)+1}{(s+1)^2} + \frac{1}{s^2(s+1)^2} - \frac{e^{-s}}{s^2(s+1)}.\end{aligned}$$

Partial fractions on the second and third terms leads to

$$\begin{aligned}Y(s) &= \left[\frac{1}{s+1} + \frac{1}{(s+1)^2}\right] + \left[-\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{(s+1)^2}\right] + e^{-s}\left[\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s+1}\right] \\ &= -\frac{2}{s} + \frac{1}{s^2} + \frac{3}{s+1} + \frac{2}{(s+1)^2} + e^{-s}\left(\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s+1}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}y(t) &= -2 + t + 3e^{-t} + 2te^{-t} + [1 - (t-1) - e^{-(t-1)}]\,h(t-1) \\ &= (3 + 2t)e^{-t} + t - 2 + (2 - t - e^{1-t})\,h(t-1).\end{aligned}$$

This solution is identical to that in Example 6.28.

We now use Laplace transforms to solve other initial-value problems with piecewise-defined and/or discontinuous nonhomogeneities. We invite the reader to make comparisons to solutions obtained by matching solutions and their derivatives at discontinuities.

Example 6.29 Find the amount of salt in the tank of Example 3.10 if pure water is added for the first ten minutes at 5 millilitres per second, and then the brine mixture is added thereafter.

Solution If we change from the letter S in Example 3.10 to represent the number of grams of salt in the tank to x, the initial-value problem for $x(t)$ is

$$\frac{dx}{dt} = 0.1h(t - 600) - \frac{5x}{10^6}, \qquad x(0) = 5000.$$

When we take Laplace transforms on both sides of the differential equation,

$$sX - 5000 = \frac{e^{-600s}}{10s} - \frac{5X}{10^6} \qquad \Longrightarrow \qquad X(s) = \frac{e^{-600s}/(10s) + 5000}{s + 5/10^6}.$$

Inverse transforms now give

$$\begin{aligned}x(t) &= 5000e^{-5t/10^6} + \frac{1}{10}\mathcal{L}^{-1}\left\{\left(\frac{10^6/5}{s} - \frac{10^6/5}{s+5/10^6}\right)e^{-600s}\right\}\\ &= 5000e^{-5t/10^6} + 20\,000\left[1 - e^{-5(t-600)/10^6}\right]h(t-600).\bullet\end{aligned}$$

A graph of this function is shown in Figure 6.30a. In agreement with Theorem 6.8, there is a discontinuity in the slope of the graph at $t = 600$ when the input rate of salt is discontinuous. Since pure water is added for the first 10 minutes, the amount of salt in the tank decreases. This is evidenced by the negative slope of the graph for $0 < t < 600$. At $t = 600$, the amount of salt in the tank is

$$\lim_{t\to 600^-} x(t) = \lim_{t\to 600^-}(5000e^{-5t/10^6}) = 5000e^{-600/20\,000} \approx 4852 \text{ grams.}$$

Because the concentration of salt in the tank at this time is $4852/10^6 = 4.852 \times 10^{-3}$ g/mL which is less than the concentration 0.02 g/mL of incoming salt, the amount of salt now begins to increase (the slope of the graph is positive). For large t, the amount of salt in the tank approaches 20 000 grams, with concentration $20\,000/10^6 = 0.02$ g/mL, the concentration of incoming brine. The asymptote is shown in Figure 6.30b.$\bullet$

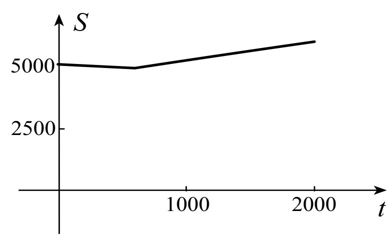


Figure 6.30a

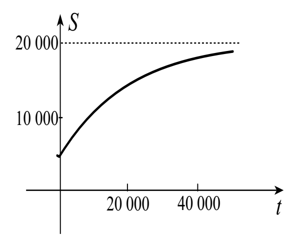


Figure 6.30b

It should be noted that Laplace transforms cannot be applied to mixing problems when the amount of liquid in the tank is not constant. For instance, the differential equation for the amount of salt in the tank of Example 3.11 is

$$\frac{dS}{dt} = \frac{1}{5} - \frac{5S}{10^6 + 5t}.$$

We have no formula for the Laplace transform of the second term on the right side of this equation.

Example 6.30 A 2-kilogram mass is suspended from a spring with constant 512 newtons per metre. It is set into motion by lifting it 10 centimetres above its equilibrium position and then releasing it. A sinusoidal force $A\sin 8t$ acts on the mass but only for $t > 1$. Find the position of the mass as a function of time if damping is negligible.

Solution The initial-value problem for displacement is

$$2\frac{d^2x}{dt^2} + 512x = A\sin 8t\, h(t-1), \qquad x(0) = \frac{1}{10}, \qquad x'(0) = 0.$$

If we take Laplace transforms,

$$2\left(s^2X - \frac{s}{10}\right) + 512X = Ae^{-s}\mathcal{L}\{\sin 8(t+1)\}$$

$$= Ae^{-s}\mathcal{L}\{\cos 8\,\sin 8t + \sin 8\,\cos 8t\}$$
$$= Ae^{-s}\left[\frac{8\cos 8}{s^2+64} + \frac{(\sin 8)s}{s^2+64}\right].$$

Hence,

$$X(s) = \frac{s}{10(s^2+256)} + \frac{Ae^{-s}[8\cos 8 + (\sin 8)s]}{2(s^2+256)(s^2+64)}.$$

Partial fractions on the second term gives

$$X(s) = \frac{s}{10(s^2+256)} + \frac{Ae^{-s}}{384}\left[\frac{8\cos 8 + (\sin 8)s}{s^2+64} - \frac{8\cos 8 + (\sin 8)s}{s^2+256}\right],$$

and therefore

$$x(t) = \frac{1}{10}\cos 16t + \frac{A}{384}\mathcal{L}^{-1}\left\{\frac{8\cos 8 + (\sin 8)s}{s^2+64} - \frac{8\cos 8 + (\sin 8)s}{s^2+256}\right\}_{t\to t-1} h(t-1)$$
$$= \frac{1}{10}\cos 16t + \frac{A}{384}\left[\cos 8\sin 8t + \sin 8\cos 8t - \frac{1}{2}\cos 8\sin 16t\right.$$
$$\left. - \sin 8\cos 16t\right]_{t\to t-1} h(t-1)$$
$$= \frac{1}{10}\cos 16t + \frac{A}{384}\big[\cos 8\,\sin 8(t-1) + \sin 8\,\cos 8(t-1)$$
$$- \frac{1}{2}\cos 8\,\sin 16(t-1) - \sin 8\,\cos 16(t-1)\big]\,h(t-1).$$

The form of the solution changes at $t = 1$ when the force $A\sin 8t$ is applied. For $0 \le t \le 1$,

$$x(t) = \frac{1}{10}\cos 16t,$$

and for $t > 1$,

$$x(t) = \frac{1}{10}\cos 16t + \frac{A}{384}\big[\cos 8\,\sin 8(t-1) + \sin 8\,\cos 8(t-1)$$
$$- \frac{1}{2}\cos 8\,\sin 16(t-1) - \sin 8\,\cos 16(t-1)\big].$$

For $0 \le t \le 1$, motion is simple harmonic with ampitude $1/10$ and period $\pi/8$. For $t > 1$, motion is periodic with period $\pi/4$, but it is not simple harmonic. The function is graphed in Figure 6.31 for $A = 100$. Even though the force is discontinuous at $t = 1$, the graph is continuous and so also is the first derivative. In other words, displacement and velocity of the mass are continuous at $t = 1$. This could be shown algebraically, but it is a direct result of Theorem 6.9. There is a discontinuity in the second derivative, but this cannot be seen graphically.•

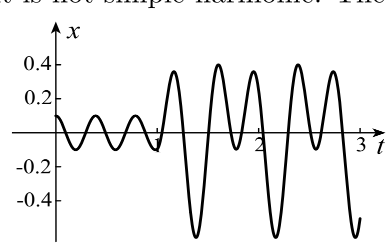


Figure 6.31

Example 6.31 Repeat Example 6.30 when the applied force is $A\sin 16t$.

Solution The initial-value problem for displacement is

$$2\frac{d^2x}{dt^2} + 512x = A\sin 16t\, h(t-1), \qquad x(0) = \frac{1}{10}, \qquad x'(0) = 0.$$

If we take Laplace transforms,

$$\begin{aligned} 2\left(s^2X - \frac{s}{10}\right) + 512X &= Ae^{-s}\mathcal{L}\{\sin 16(t+1)\} \\ &= Ae^{-s}\mathcal{L}\{\cos 16\, \sin 16t + \sin 16\, \cos 16t\} \\ &= Ae^{-s}\left[\frac{16\cos 16}{s^2+256} + \frac{(\sin 16)s}{s^2+256}\right]. \end{aligned}$$

Hence,

$$X(s) = \frac{s}{10(s^2+256)} + \frac{Ae^{-s}[16\cos 16 + (\sin 16)s]}{2(s^2+256)^2}.$$

First we use Table 6.2 to calculate that

$$\mathcal{L}^{-1}\left\{\frac{16\cos 16}{(s^2+256)^2} + \frac{(\sin 16)s}{(s^2+256)^2}\right\} = \frac{16\cos 16}{2(16^3)}(\sin 16t - 16t\cos 16t) + \frac{\sin 16}{32}t\sin 16t.$$

Hence,

$$\begin{aligned} x(t) = \frac{1}{10}\cos 16t + \frac{A}{2}\bigg\{&\frac{\cos 16}{512}[\sin 16(t-1) - 16(t-1)\cos 16(t-1)] \\ &+ \frac{\sin 16}{32}(t-1)\sin 16(t-1)\bigg\}h(t-1). \end{aligned}$$

Because of the $(t-1)$-factors, we have undamped resonance. It is interesting to plot a graph of this function. Figure 6.32 is a plot on the interval $0 \le t \le 4$ (with $A = 2$). It looks like damped oscillations as opposed to resonance. Figure 6.33, with a longer time interval, shows the resonance.●

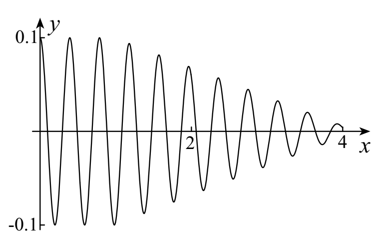


Figure 6.32

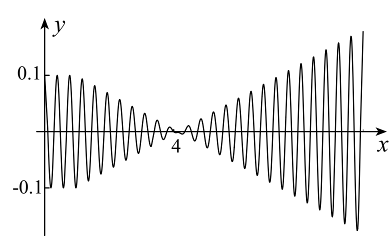


Figure 6.33

The delayed sinusoidal nonhomogeneity presented no problem in Examples 6.30 and 6.31. When the nonhomogeneity is periodic, but not sinusoidal, additional difficulties arise. Compared to a solution by methods of Chapter 4, however, Laplace transforms are vastly superior. We illustrate in the following two examples.

Example 6.32 Solve the initial-value problem

$$x'' + 4x = f(t), \qquad x(0) = 0, \qquad x'(0) = 0,$$

where $f(t)$ is the periodic function

$$f(t) = \begin{cases} 1, & 0 < t < \pi/2 \\ 0, & \pi/2 < t < \pi \end{cases} \qquad f(t+\pi) = f(t).$$

Solution When we take Laplace transforms of both sides of the differential equation, and use property 6.16 for the transform of a periodic function, we obtain

$$\begin{aligned} s^2X + 4X = \mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-\pi s}} \int_0^{\pi/2} e^{-st}\,dt = \frac{1}{1-e^{-\pi s}} \mathcal{L}\{h(t) - h(t-\pi/2)\} \\ &= \frac{1}{(1+e^{-\pi s/2})(1-e^{-\pi s/2})}\left(\frac{1}{s} - \frac{e^{-\pi s/2}}{s}\right) = \frac{1}{s(1+e^{-\pi s/2})}. \end{aligned}$$

Thus,

$$X(s) = \frac{1}{s(s^2+4)(1+e^{-\pi s/2})}.$$

We now partial fraction $1/[s(s^2+4)]$, and expand $1/(1+e^{-\pi s/2})$ in a geometric series

$$X(s) = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\left(1 - e^{-\pi s/2} + e^{-\pi s} - e^{-3\pi s/2} + \cdots\right).$$

Each term in the series has an easily calculated inverse transform,

$$x(t) = \frac{1}{4}(1-\cos 2t) - \frac{1}{4}[1-\cos 2(t-\pi/2)]h(t-\pi/2) + \frac{1}{4}[1-\cos 2(t-\pi)]h(t-\pi) - \cdots .$$

In sigma notation,

$$x(t) = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n[1-\cos 2(t-n\pi/2)]\,h(t-n\pi/2).$$

To evaluate $x(t)$ for any given t, it is necessary to include only those terms in the series for which $n\pi/2 < t$. For example, the solution at $t = 3.4$ is given by

$$x(3.4) = \frac{1}{4}[1-\cos 2(3.4)] - \frac{1}{4}[1-\cos 2(3.4-\pi/2)] + \frac{1}{4}[1-\cos 2(3.4-\pi)] = -0.402. \bullet$$

Two points are noteworthy in this example. First, consider using the techniques of Chapter 4 to find $x(3.4)$. We would solve the differential equation on the intervals $0 < t < \pi/2$, $\pi/2 < t < \pi$, $\pi < t < 3\pi/2$, match at $t = \pi/2$ and $t = \pi$, and then find $x(3.4)$ from the solution for $\pi < t < 3\pi/2$. Try it. You will be convinced that Laplace transforms are superior. Secondly, recall that after Example 5.10 in Section 5.3, we questioned whether non-sinusoidal, periodic forces could produce resonance in vibrating mass-spring systems. If we interpret this problem as describing oscillations $x(t)$ of a 1 kilogram mass on the end of a spring with constant 4 newtons per metre, there is resonance, but it is not obvious. If we write out the first few terms of the series for $x(t)$, we obtain

$$\begin{aligned} x(t) = \frac{1}{4}\big\{&(1-\cos 2t)h(t) - [1-\cos 2(t-\pi/2)]h(t-\pi/2) + [1-\cos 2(t-\pi)]h(t-\pi) \\ &+ [1-\cos 2(t-3\pi/2)]h(t-3\pi/2) + [1-\cos 2(t-2\pi)]h(t-2\pi) \\ &+ [1-\cos 2(t-5\pi/2)]h(t-5\pi/2) + [1-\cos 2(t-3\pi)]h(t-3\pi) + \cdots\big\} \end{aligned}$$

$$= \frac{1}{4}\big[1 - \cos 2t - (1+\cos 2t)h(t-\pi/2) + (1-\cos 2t)h(t-\pi)$$
$$- (1+\cos 2t)h(t-3\pi/2) + (1-\cos 2t)h(t-2\pi)$$
$$- (1+\cos 2t)h(t-5\pi/2) + (1-\cos 2t)h(t-3\pi)\cdots\big]$$

$$= \frac{1}{4}\begin{cases} 1-\cos 2t, & 0 \le t < \pi/2 \\ -2\cos 2t, & \pi/2 \le t < \pi \\ 1-3\cos 2t, & \pi \le t < 3\pi/2 \\ -4\cos 2t, & 3\pi/2 \le t < 2\pi \\ 1-5\cos 2t, & 2\pi \le t < 5\pi/2 \\ -6\cos 2t, & 5\pi/2 \le t < 3\pi \\ \text{etc.} \end{cases}$$

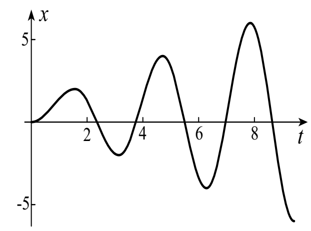


Figure 6.34

Amplitudes of these oscillations increase with time. We have shown a graph of the function in Figure 6.34. Resonance has occurred because the natural frequencey of the mass-spring system is $1/\pi$, and this is the frequency of the applied force $f(t)$.

The following example is fascinating. It may defy your intuition at first, but when we reason it out, it makes perfect sense.

Example 6.33 A mass M hangs motionless from a spring with constant k. At time $t = 0$, the mass is acted upon by the periodic force in Figure 6.35. The force is a constant value F for p seconds, then it is turned off for p seconds, back on for p seconds, and so on. During its subsequent motion, the mass experiences no damping. Find displacements of the mass Discuss these displacements when $p = 2\pi\sqrt{M/k}$.

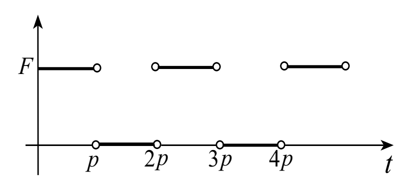


Figure 6.35

Solution The initial-value problem for displacements of the mass is

$$M\frac{d^2x}{dt^2} + kx = f(t), \quad x(0) = 0. \quad x'(0) = 0,$$

where $f(t)$ is the function in Figure 6.35. Its representation in terms of Heaviside functions is

$$f(t) = F[h(t) - h(t-p)], \quad 0 < t < 2p, \qquad f(t+2p) = f(t).$$

When we take Laplace transforms,

$$Ms^2X + kX = \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2ps}}\mathcal{L}\{F[1-h(t-p)]\} = \frac{F}{1-e^{-2ps}}\left(\frac{1}{s} - \frac{e^{-ps}}{s}\right).$$
$$= \frac{F(1-e^{-ps})}{s(1+e^{-ps})(1-e^{-ps})} = \frac{F}{s(1+e^{-ps})}.$$

Hence,

$$X(s) = \frac{F}{s(Ms^2+k)(1+e^{-ps})}.$$

With the partial fraction decomposition of $1/[s(Ms^2+k)]$, and geometric series for $1/(1+e^{-ps})$, the inverse transform of $X(s)$ is

$$\begin{aligned} x(t) &= F\mathcal{L}^{-1}\left\{\left(\frac{1/k}{s} - \frac{Ms/k}{Ms^2+k}\right)\frac{1}{1+e^{-ps}}\right\} \\ &= \frac{F}{k}\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+k/M}\right)\sum_{n=0}^{\infty}(-1)^n e^{-pns}\right\} \\ &= \frac{F}{k}\sum_{n=0}^{\infty}(-1)^n\left[1-\cos\sqrt{\frac{k}{M}}(t-pn)\right]h(t-pn). \end{aligned}$$

When $p = 2\pi\sqrt{M/k}$, displacements become

$$\begin{aligned} x(t) &= \frac{F}{k}\sum_{n=0}^{\infty}(-1)^n\left[1-\cos\frac{2\pi}{p}(t-pn)\right]h(t-pn) \\ &= \frac{F}{k}\sum_{n=0}^{\infty}(-1)^n\left[1-\cos\frac{2\pi t}{p}\right]h(t-np) \\ &= \frac{F}{k}\left(1-\cos\frac{2\pi t}{p}\right)\sum_{n=0}^{\infty}(-1)^n h(t-np). \end{aligned}$$

To graph this function, we write the summation out,

$$x(t) = \frac{F}{k}\left(1-\cos\frac{2\pi t}{p}\right)[h(t)-h(t-p)+h(t-2p)-\cdots],$$

in which case we see that

$$x(t) = \begin{cases} \dfrac{F}{k}\left(1-\cos\dfrac{2\pi t}{p}\right), & 0 \le t < p \\ 0, & p \le t < 2p \\ \dfrac{F}{k}\left(1-\cos\dfrac{2\pi t}{p}\right), & 2p \le t < 3p \\ 0, & 3p \le t < 4p \\ \text{etc.} \end{cases}$$

A graph of this function is shown in Figure 6.36. Here is an explanation of why this happens. During the interval $0 \le t \le p/2$, the force F moves the mass upward against the spring, and for $p/2 < t \le p$, the spring returns the mass against F to the equilibrium position. When the mass reaches equilibrium, its velocity is zero. Since the force now becomes zero, the mass remains at equilibrium for time p. This sequence of motion then repeats itself in each interval of length $2p$ thereafter.●

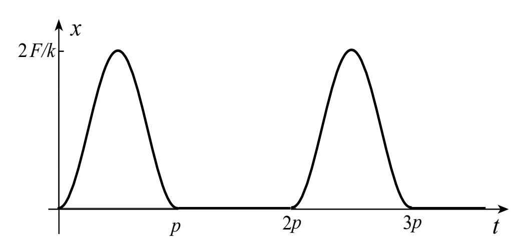


Figure 6.36

EXERCISES 6.4

1. Solve the initial-value problem

$$\frac{dy}{dt} + 3y = f(t), \qquad y(0) = 1,$$

where $\begin{cases} t, & 0 < t < 1 \\ 1, & t > 1. \end{cases}$ by:
(a) the techniques of Chapter 4,
(b) Laplace transforms.

In Exercises 2–12 solve the initial-value problem.

2. $y'' + 9y = f(t), \quad y(0) = 1, \quad y'(0) = 2, \qquad \text{where } f(t) = \begin{cases} 0, & 0 < t < 4 \\ 1, & t > 4 \end{cases}$

3. $y'' + 9y = f(t), \quad y(0) = 1, \quad y'(0) = 2, \qquad \text{where } f(t) = \begin{cases} 2, & 0 < t < 4 \\ 0, & t > 4 \end{cases}$

4. $y'' + 4y' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = -1, \qquad \text{where } f(t) = \begin{cases} t, & 0 < t < 1 \\ 1, & t > 1 \end{cases}$

5. $y'' + 4y' + 4y = f(t), \quad y(0) = -1, \quad y'(0) = 0, \qquad \text{where } f(t) = \begin{cases} 2 - t, & 0 < t < 2 \\ t - 2, & t > 2 \end{cases}$

6. $y'' + 4y' + 3y = f(t), \quad y(0) = 1, \quad y'(0) = 2, \qquad \text{where } f(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

7. $y'' + 4y' + 3y = f(t), \quad y(0) = 1, \quad y'(0) = 2, \qquad \text{where } f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

8. $y'' + 2y' + 5y = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad \text{where } f(t) = \begin{cases} 3, & 0 < t < 1 \\ -3, & t > 1 \end{cases}$

9. $y'' + 2y' + 5y = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad \text{where } f(t) = \begin{cases} 4, & 0 < t < 1 \\ -4, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

10. $y'' + 16y = f(t), \quad y(0) = 2, \quad y'(0) = 0, \qquad \text{where } f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$

11. $y'' + 16y = f(t), \quad y(0) = 2, \quad y'(0) = 0, \quad \text{where } f(t) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$

12. $y'' + y' = f(t), \quad y(0) = 0, \quad y'(0) = 0, \quad \text{where } f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$

13. Use Laplace transforms to solve the mixing problem of Example 3.10 in Section 3.4.

14. Can you use Laplace transforms to solve the mixing problem in Example 3.11 of Section 3.4?

15. Solve Example 3.10 in Section 3.4 if after 10 minutes the concentration of the brine being added to the tank changes to 1 kilogram per 100 litres.

16. Find the amount of salt in the tank of Example 3.10 in Section 3.4 if the brine mixture is added for 2 minutes, replaced by pure water for 2 minutes, replaced by the brine mixture for 2 minutes, etc.

17. When a patient is admitted to the hospital, the amount of glucose in his bloodstream is g_0 grams. He is immediately put on intravenous which transfers glucose to his bloodstream at a rate of R grams per minute. At any given time, his body uses the glucose up a rate prportional to how much is present in the bloodstream at that time. If he remains on intravenous for 4 hours and then the intravenous is discontinued, how much glucose is in the bloodstream 6 hours after he is admitted?

18. Find the amount of glucose in the bloodstream of the patient in Exercise 17 as a function of time (in minutes) after he is admitted to the hospital if glucose is administered for an hour, turned off for an hour, turned on for an hour, turned off for an hour, etc.

19. The initial mass of a certain species of fish in a lake is estimated as m_0 kilograms. Suppose that left alone, the fish would increase their mass at a rate described by the Malthusian model 3.11. Commercial fishing harvests (removes) H kilograms each year, at a uniform rate, but only in the first month of the year.
(a) Find the mass $m(t)$ of fish in the lake as a function of time t.
(b) Determine the value of H in order that the mass of fish in the lake return to m_0 after one year.

20. A 100-gram mass is suspended from a spring with constant 40 newtons per metre. The mass is pulled 10 centimetres above its equilibrium position and given velocity 2 metres per second downward. If a force of 100 newtons acts vertically upward for the first 4 seconds, find the position of the mass as a function of time. Ignore all damping.

21. Repeat Exercise 20 if the force is turned on after 4 seconds.

22. Repeat Exercise 20 if a damping force with constant $\beta = 5$ also acts on the mass.

23. Repeat Exercise 21 if a damping force with constant $\beta = 5$ also acts on the mass.

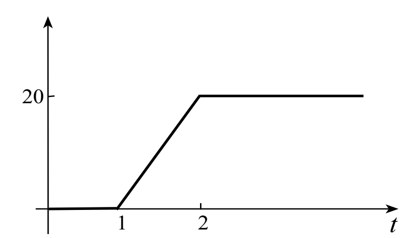
24. Repeat Exercise 20 if a damping force with constant $\beta = 1$ also acts on the mass.

25. Repeat Exercise 21 if a damping force with constant $\beta = 1$ also acts on the mass.

26. Repeat Example 6.33 if $p = 4\pi\sqrt{M/k}$.

27. Repeat Example 6.33 if $p = 3\pi\sqrt{M/k}$.

28. A 1 kilogram mass is motionless at the end of a spring with constant 16 newtons per metre. When the ramp force in the figure to the right (units of newtons) acts the mass, the mass moves vertically without damping. Find its subsequent displacements. Plot a graph of the displacement function.



6.5 The Dirac Delta Function and its Applications

There are many common situations that cannot be represented mathematically by functions as we know them. For instance, consider:

1. suddenly adding a sizeable quantity of dissolving substance in a mixing problem in Section 3.4
2. striking a mass on the end of a spring with a hammer in Section 5.3
3. accommodating a voltage spike in an electric circuit in Section 5.4
4. representing a concentrated load on a beam in Section 5.5

We introduce the "Dirac delta" function in this section in order to model these situations mathematically. We begin with what are called *unit pulse functions*. The **unit pulse** at time $t = t_0$ of duration a is the function in Figure 6.37. It can be represented in terms of Heaviside unit step functions as

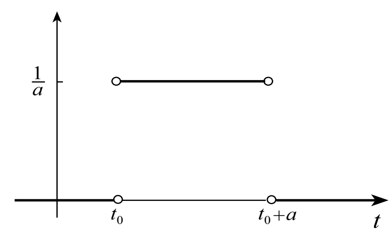


Figure 6.37

$$p(t_0, a, t) = \frac{1}{a}[h(t - t_0) - h(t - t_0 - a)]. \tag{6.26}$$

Value t_0 identifies the time at which the pulse begins, and a is the duration of the pulse. What is important to notice is that the area under the curve is one; hence the name *unit* pulse. The units of $p(t_0, a, t)$ are inverses of the units of t. If t is time in seconds (s), then units of $p(t_0, a, t)$ are s^{-1}. The Laplace transform of this function is

$$\mathcal{L}\{p(t_0, a, t)\} = \frac{1}{as}\left[e^{-t_0 s} - e^{-(t_0+a)s}\right]. \tag{6.27}$$

The Unit Impulse

Even more important than the unit pulse is the *unit impulse*. It is the limit of the unit pulse $p(t_0, a, t)$ as the time interval $t_0 < t < t_0 + a$ becomes indefinitely short. As a gets smaller and smaller in Figure 6.37, the area under the curve remains unity; the function is nonzero over shorter and shorter time intervals, but the height of the function gets larger and larger. We have shown the situation for $a = 1/10$, $1/20$, $1/40$, and $1/80$ in Figure 6.38.

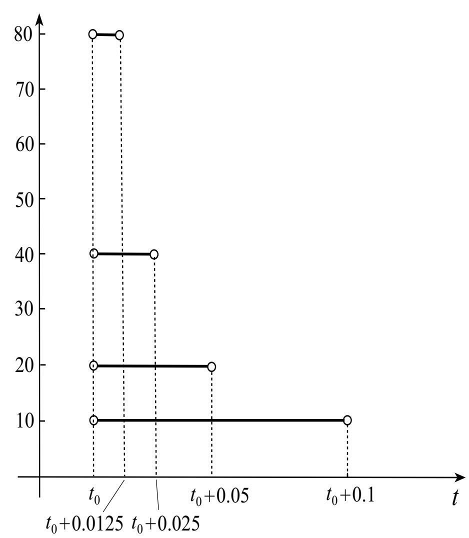


Figure 6.38

The limit of this function as $a \to 0$ is not a function in the normal sense of function. It has value 0 for all t except $t = t_0$ where its value is "infinite". Such functions are discussed in advanced mathematics; they are known as *generalized functions* or *distributions*. This particular one is called the **unit impulse** or the **Dirac delta** function. It is denoted by

$$\delta(t - t_0) = \lim_{a \to 0} \frac{1}{a}[h(t - t_0) - h(t - t_0 - a)]. \tag{6.28}$$

The Dirac delta function can be defined as the limit of other sequences of functions besides those in Figure 6.38; all lead to identical properties. Two such sequences are shown in Figures 6.39 and 6.40. In both cases, the area under each curve is unity. Like the functions in Figure 6.38, those in Figure 6.39 are discontinuous, but they are symmetric around t_0. The functions in Figure 6.40 are continuous and symmetric around t_0.

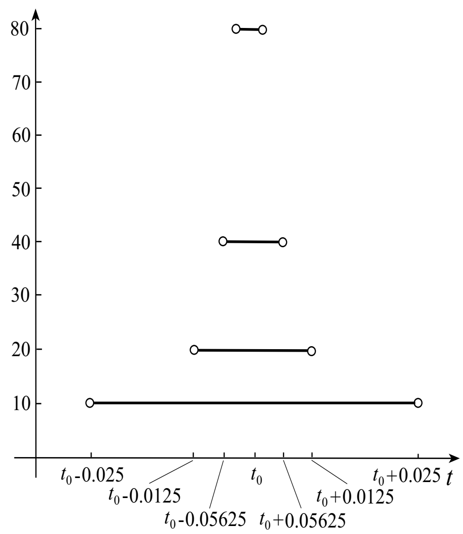


Figure 6.39

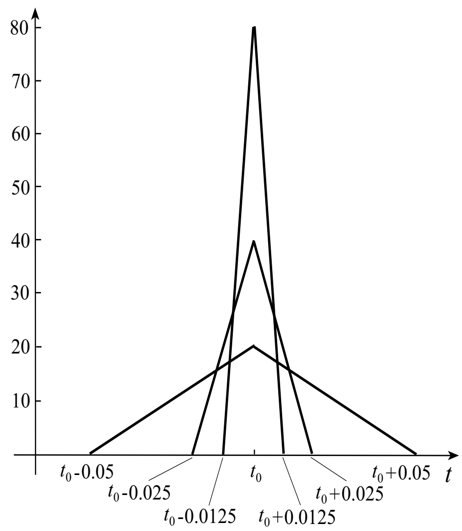


Figure 6.40

The Dirac delta function does not conform to the conditions of Theorem 6.1; it is of exponential order, but it is not piecewise-continuous on every finite interval. It does, however, have a Laplace transform, and we could write

$$\mathcal{L}\{\delta(t - t_0)\} = \mathcal{L}\left\{\lim_{a \to 0} p(t_0, a, t)\right\}.$$

We define the Laplace transform of $\delta(t - t_0)$ by interchanging the operations of taking the limit and taking the Laplace transform; that is, we define

$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{a \to 0} \mathcal{L}\{p(t_0, a, t)\}. \tag{6.29}$$

Substituting from equation 6.27 and using L'Hopital's rule on the limit gives

$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{a \to 0}\left(\frac{e^{-t_0 s} - e^{-(t_0+a)s}}{as}\right) = \lim_{a \to 0}\left(\frac{se^{-(t_0+a)s}}{s}\right) = e^{-t_0 s}. \tag{6.30}$$

For readers who find this definition unsatisfactory, an alternative is contained in the Appendix at the end of this section. It briefly discusses how "generalized functions" such as the delta function should be manipulated.

Notice that when $t_0 = 0$, the Laplace transform of $\delta(t)$ is

$$\mathcal{L}\{\delta(t)\} = 1.$$

In Section 6.1, we proved that the limit as $s \to \infty$ of the Laplace transform of every piecewise continuous function of exponential order is equal to zero. The limit of the Laplace transform of $\delta(t)$ is not zero as $s \to \infty$, but it is not a piecewise-continuous function.

We will give arguments to demonstrate that the delta function can be used to model the situations described at the beginning of this section, situations that cannot be modelled by ordinary functions. Important to realize is the units of $\delta(t-t_0)$. Since the delta function is the limit of the unit pulse function, and taking limits does not alter units, units of the delta function are those of the pulse function, namely, 1 over the units of t. Hence, if t is time in seconds (s), then units of $\delta(t-t_0)$ are s^{-1}.

We begin with a mixing problem.

Mixing Problems and the Delta Function

A tank contains 1000 litres of water in which 5 kilograms of salt have been dissolved. A brine mixture with concentration 2 kilograms of salt for each 100 litres of water is added to the tank at 5 millilitres per second. At the same time, mixture is being drawn from the bottom of the tank at 5 millilitres per second. Suppose that 3 kilograms of salt are suddenly added to the tank at the 5 minute mark, and we wish to find the amount of salt in the tank as a function of time. As usual, we assume that the mixture in the tank is always sufficiently well-stirred that at any given time, concentration of salt is the same at all points in the tank, even when the 3 kilograms is suddenly added to the tank. This amounts to assuming that the 3 kilograms of salt dissolve instantaneously.

If we let $x(t)$ represent the number of grams of salt in the tank at any given time, then $x(0) = 5000$. As in Section 3.4, we write symbolically that

$$\frac{dx}{dt} = \begin{pmatrix} \text{rate at which} \\ \text{salt is added} \end{pmatrix} - \begin{pmatrix} \text{rate at which} \\ \text{salt is removed} \end{pmatrix}.$$

Since 5 mL of mixture enter the tank each second, and each millilitre contains 0.02 g of salt, it follows that salt is being added to the tank at a constant rate of 0.1 g/s. The rate at which salt is removed from the tank is not constant; it changes as the concentration of salt in the tank changes. Since the tank always contains 10^6 mL of solution, the concentration of salt in the solution at time t, in grams per millilitre, is $x(t)/10^6$. With solution being drawn off at 5 mL/s, the rate at which salt leaves the tank in grams per second is $5x/10^6$. The difficulty is how to model the sudden insertion of 3 kilograms of salt into the tank at the 5 minute mark. We will show that the delta function is the key. First, suppose we assume that the 3 kilograms of salt are added to the tank uniformly over a time interval of a seconds beginning at time $t = 5$. This means adding it a rate of $3000/a$ g/s over the a seconds. This can be represented by

$$\frac{3000}{a}[h(t-300) - h(t-300-a)].$$

If we now insert these rates into the equation for dx/dt, we obtain

$$\frac{dx}{dt} = \frac{1}{10} - \frac{5x}{10^6} + \frac{3000}{a}[h(t-300) - h(t-300-a)].$$

When we take Laplace transforms,

$$sX - 5000 = \frac{1}{10s} - \frac{5X}{10^6} + \frac{3000}{as}[e^{-300s} - e^{-(300+a)s}],$$

from which

$$\begin{aligned} X(s) &= \frac{5000 + \dfrac{1}{10s} + \dfrac{3000}{as}[e^{-300s} - e^{-(300+a)s}]}{s + \dfrac{5}{10^6}} \\ &= \frac{5000}{s+5/10^6} + \frac{10^5}{5}\left(\frac{1}{s} - \frac{1}{s+5/10^6}\right) \\ &\quad + \frac{6\times 10^8}{a}\left(\frac{1}{s} - \frac{1}{s+5/10^6}\right)[e^{-300s} - e^{-(300+a)s}]. \end{aligned}$$

Inverse transforms give

$$\begin{aligned} x(t) = 5000e^{-5t/10^6} &+ \frac{10^5}{5}(1 - e^{-5t/10^6}) \\ &+ (6\times 10^8)\left\{\frac{\left[1 - e^{-5(t-300)/10^6}\right]h(t-300) - \left[1 - e^{-5(t-300-a)/10^6}\right]h(t-300-a)}{a}\right\}. \end{aligned}$$

This is the number of grams of salt in the tank if the 3 kilograms of salt are added at a constant rate over a seconds beginning at $t = 300$. In order to find the result when the 3 kilograms of salt is added instantaneously, we use L'Hopital's rule to take the limit of this function as $a \to 0$,

$$\begin{aligned} x(t) &= 20\,000 - 15,000e^{-5t/10^6} + (6\times 10^8)\lim_{a\to 0}[(5/10^6)e^{-5(t-300-a)/10^6}h(t-300-a)] \\ &= 20\,000 - 15,000e^{-5t/10^6} + 3000e^{-5(t-300)/10^6}h(t-300). \end{aligned}$$

We now show that when the delta function is used to model the instantaneous addition of the 3 kilograms of salt, we get the same result with a fraction of the work. Since the units of $\delta(t-t_0)$ are seconds to the negative 1, $3000\delta(t-300)$ has units of grams per second, consistent with other input and output rates. Suppose we use this expression to represent the addition of the 3000 grams of salt at $t = 300$ seconds. The differential equation for $x(t)$ becomes

$$\frac{dx}{dt} = \frac{1}{10} - \frac{5x}{10^6} + 3000\delta(t-300).$$

When we take Laplace transforms,

$$sX - 5000 = \frac{1}{10s} - \frac{5X}{10^6} + 3000e^{-300s},$$

from which

$$X(s) = \frac{5000}{s+5/10^6} + \frac{1/(10s) + 3000e^{-300s}}{s+5/10^6}.$$

Inverse transforms give

$$\begin{aligned}x(t) &= 5000e^{-5t/10^6} + \frac{10^5}{5}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+5/10^6}\right\} + 3000e^{-5(t-300)/10^6}h(t-300)\\ &= 20\,000 - 15\,000e^{-5t/10^6} + 3000e^{-5(t-300)/10^6}h(t-300).\end{aligned}$$

We have shown therefore that the delta function representation of the instantaneous addition of 3 kilograms of salt yields the same result as adding the salt over a small interval of time and then taking the limit as this interval approaches zero, and it does so with far less work. It is worthwhile noting that

$$\lim_{t\to 300^-} x(t) = 20\,000 - 15\,000e^{-5(300)/10^6},$$
$$\lim_{t\to 300^+} x(t) = 20\,000 - 15\,000e^{-5(300)/10^6} + 3000.$$

In other words, the amount of salt jumps by 3000 grams at $t = 300$ seconds, as we should expect.

Displacements in Mass-Spring Systems and the Delta Function

We now consider the situation in which a pulse represents an external force applied to the mass in a vibrating mass-spring system such as that in Figure 6.41. When damping and surface friction are negligible, the differential equation describing the position of the mass relative to its equilibrium position is $M\,d^2x/dt^2 + kx = f(t)$ where $f(t)$ represents all forces on M other than the spring.

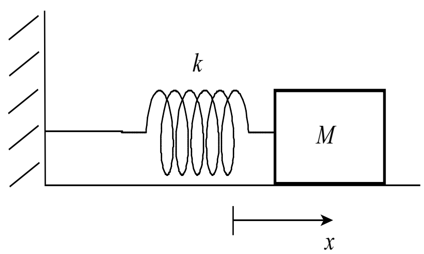


Figure 6.41

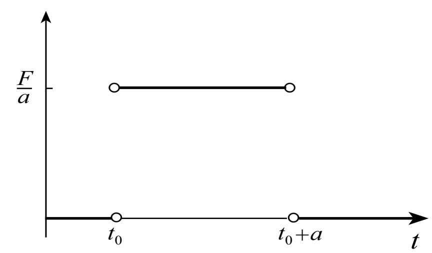


Figure 6.42

Suppose the mass is at rest at its equilibrium position for t_0 seconds starting at time $t = 0$, and at time t_0, the mass is subjected to the pulse in Figure 6.42. The magnitude of the force is F/a, but the area under the curve is F, and is so for any length of duration a of the pulse. We call this a pulse of size F. The initial-value problem for displacements of the mass due to this pulse is

$$M\frac{d^2x}{dt^2} + kx = F\,p(t_0, a, t), \qquad x(0) = 0, \qquad x'(0) = 0. \tag{6.31}$$

If we take Laplace transforms of both sides of the differential equation, and use formula 6.27,

$$Ms^2X + kX = \frac{F}{as}\left[e^{-t_0 s} - e^{-(t_0+a)s}\right] \quad\Longrightarrow\quad X(s) = \frac{F[e^{-t_0 s} - e^{-(t_0+a)s}]}{as(Ms^2+k)}.$$

Partial fractions give

$$X(s) = \frac{F}{ka}\left(\frac{1}{s} - \frac{s}{s^2 + k/M}\right)(e^{-t_0 s} - e^{-(t_0+a)s}),$$

from which

$$x(t) = \frac{F}{ka}\left[1 - \cos\sqrt{\frac{k}{M}}(t - t_0)\right] h(t - t_0)$$
$$- \frac{F}{ka}\left[1 - \cos\sqrt{\frac{k}{M}}(t - t_0 - a)\right] h(t - t_0 - a). \qquad (6.32)$$

A typical graph of this function is shown in Figure 6.43. The mass is at equilibrium for $0 \le t < t_0$. At time t_0, the force causes the mass to move. While the force acts ($t_0 < t < t_0 + a$), the mass experiences simple harmonic motion with amplitude $F/(ka)$,

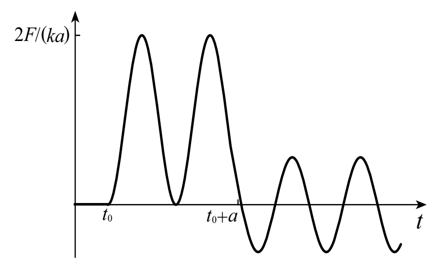


Figure 6.43

$$x(t) = \frac{F}{ka}\left[1 - \cos\sqrt{\frac{k}{M}}(t - t_0)\right]. \qquad (6.33a)$$

After the force is removed at $t = t_0 + a$, the displacement of the mass is

$$x(t) = \frac{F}{ka}\left[\cos\sqrt{\frac{k}{M}}(t - t_0 - a) - \cos\sqrt{\frac{k}{M}}(t - t_0)\right]. \qquad (6.33b)$$

This is once again simple harmonic motion, but with a different amplitude. For some values of the parameters, the amplitude may be enhanced and for other values, it may be diminished (see Exercise 15). The derivative of this function is the velocity of the mass. Since the derivative of the Heaviside function is zero, except at t_0 where it is undefined, the velocity is

$$v(t) = \frac{F}{ka}\left[\sqrt{\frac{k}{M}}\sin\sqrt{\frac{k}{M}}(t - t_0)\right] h(t - t_0)$$
$$- \frac{F}{ka}\left[\sqrt{\frac{k}{M}}\sin\sqrt{\frac{k}{M}}(t - t_0 - a)\right] h(t - t_0 - a)$$
$$= \frac{F}{a\sqrt{kM}}\left[\sin\sqrt{\frac{k}{M}}(t - t_0)h(t - t_0) - \sin\sqrt{\frac{k}{M}}(t - t_0 - a)h(t - t_0 - a)\right].$$

Left and right limits of this function are the same at t_0 and $t_0 + a$ so that there is no abrupt change in velocity at t_0 and $t_0 + a$ where the pulse is discontinuous. This is reflected by the smoothness of the graph in Figure 6.43; the slope of the tangent line is continuous at t_0 and $t_0 + a$. This is consistent with the fact that the solution of the differential equation, and its first derivative, must be continuous even with a piecewise-continuous nonhomogeneity (Theorem 6.9).

Suppose we use L'Hopital's rule to take the limit of this displacement function as the duration time of the pulse becomes indefinitely small,

$$\begin{aligned}
x(t) &= \frac{F}{k} \lim_{a\to 0^+} \frac{\left[1-\cos\sqrt{\frac{k}{M}}(t-t_0)\right]h(t-t_0) - \left[1-\cos\sqrt{\frac{k}{M}}(t-t_0-a)\right]h(t-t_0-a)}{a} \\
&= \frac{F}{k} \lim_{a\to 0^+} \frac{\left[\sqrt{\frac{k}{M}}\sin\sqrt{\frac{k}{M}}(t-t_0-a)\right]h(t-t_0-a)}{1} \\
&= \frac{F}{\sqrt{kM}} \sin\sqrt{\frac{k}{M}}(t-t_0)h(t-t_0). \qquad (6.34)
\end{aligned}$$

This is the displacement of the mass at time t due to a pulse of size F as the duration of the pulse becomes indefinitely short. We regard this displacement as that due to an instantaneously applied force at time t_0. A typical graph of this function is shown in Figure 6.44. The mass is at equilibrium for $0 \le t < t_0$. At time t_0, the instantaneous force imparts a velocity to the mass. To find the velocity we differentiate the displacement with respect to t,

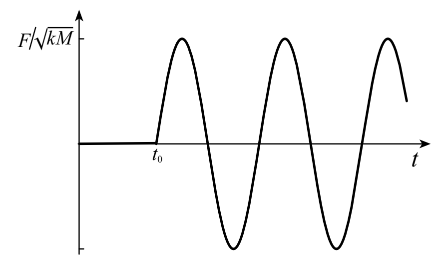


Figure 6.44

$$v(t) = \frac{F}{M}\cos\sqrt{\frac{k}{M}}(t-t_0)h(t-t_0).$$

This function is discontinuous at t_0, and its limit as $t \to t_0^+$ is F/M. In other words, the instantaneous force has given the mass an initial velocity F/M metres per second. This is reflected in the corner of the curve in Figure 6.44 at $t = t_0$, the slope of the graph jumps from zero to F/M.

Our hope is that the Dirac delta function will be the mathematical representation of an instantaneously applied force (such as that in the above mass-spring system, perhaps the result of hitting the mass with a hammer). Consider again, then, the mass-spring system in Figure 6.41. We showed that the position of the mass due to a pulse of magnitude F applied to the mass at time t_0 over a time interval of length a is given by equation 6.32. We took the limit of this function as $a \to 0$ to obtain displacement 6.34 due to an instantaneously applied force.

Suppose we replace the pulse function in equation 6.31 with $F\delta(t-t_0)$, so that the initial-value problem for displacements is

$$M\frac{d^2x}{dt^2} + kx = F\delta(t-t_0), \quad x(0) = 0, \quad x'(0) = 0. \qquad (6.35)$$

When we take Laplace transforms, we get

$$M[s^2X] + kX = Fe^{-t_0 s} \qquad \Longrightarrow \qquad X(s) = \frac{Fe^{-t_0 s}}{Ms^2+k} = \frac{F}{M}\frac{e^{-t_0 s}}{s^2+k/M}.$$

The inverse transform gives

$$x(t) = \frac{F}{M}\sqrt{\frac{M}{k}}\sin\sqrt{\frac{k}{M}}(t-t_0)h(t-t_0) = \frac{F}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}(t-t_0)h(t-t_0).$$

This is solution 6.34. We have shown then that the problem of finding displacements for a vibrating mass on the end of a spring, subjected to a pulse of magnitude F, and taking the limit as the interval of application of the pulse approaches zero, can be handled much more efficiently with the Dirac delta function. This is a partial justification for the use of the Dirac delta function to represent instantaneously applied forces to masses in mass-spring systems.

We call $F\delta(t-t_0)$ an **impulse force** of size F at t_0. Although an impulse force of size F can be thought of as the limit of a pulse of size F as the duration of the pulse approaches zero, there is a better way to view it. As a force, $F\delta(t-t_0)$ has units of newtons or kilogram-metres per second squared. Since the units of the delta function are seconds to the power negative one, it follows that F itself is not a force; it does not have units of newtons. It has units of kilogram-metres per second;, units of momentum. In other words, when an instantaneously applied force in the form $F\delta(t-t_0)$ acts on a mass, it imparts F units of momentum to the mass. This is consistent with what we saw earlier. Application of an impulse force $F\delta(t-t_0)$ to a mass M results in a velocity change of F/M. This means a change of F in momentum. Thus, instead of sayig that a vibrating mass is struck with an impulse force of size F, it is more informative to say that it is struck with a force that gives the mass F units of momentum.

Here is another example, one that includes damping.

Example 6.34 A 100-gram mass is suspended from a spring with constant 50 newtons per metre. It is set into motion by raising it 10 centimetres above its equilibrium position and giving it a velocity of 1 metre per second downward. During the subsequent motion a damping force acts on the mass and the magnitude of this force is one-fifth the velocity of the mass. If an impulse force that imparts two units of momentum to the mass is applied vertically upward to the mass at $t = 3$ seconds, find the position of the mass for all time.

Solution The initial-value problem for the position of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{1}{5}\frac{dx}{dt} + 50x = 2\delta(t-3), \qquad x(0) = \frac{1}{10}, \qquad x'(0) = -1.$$

If we multiply the differential equation by 10, and take Laplace transforms,

$$\left(s^2X - \frac{s}{10} + 1\right) + 2\left(sX - \frac{1}{10}\right) + 500X = 20e^{-3s}.$$

Thus,

$$\begin{aligned} X(s) &= \frac{s/10 - 4/5}{s^2+2s+500} + \frac{20e^{-3s}}{s^2+2s+500} \\ &= \frac{1}{10}\left[\frac{(s+1)-9}{(s+1)^2+499}\right] + \frac{20e^{-3s}}{(s+1)^2+499}. \end{aligned}$$

The inverse transform is

$$x(t) = \frac{1}{10}e^{-t}\mathcal{L}^{-1}\left\{\frac{s-9}{s^2+499}\right\} + \mathcal{L}^{-1}\left\{\frac{20e^{-3s}}{(s+1)^2+499}\right\}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+499}\right\} = 20e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+499}\right\} = \frac{20}{\sqrt{499}}e^{-t}\sin\sqrt{499}t,$$

it follows that

$$x(t) = \frac{1}{10}e^{-t}\left(\cos\sqrt{499}t - \frac{9}{\sqrt{499}}\sin\sqrt{499}t\right) + \frac{20}{\sqrt{499}}e^{-(t-3)}\sin\sqrt{499}(t-3)\,h(t-3).$$

It is straightforward to show that this solution satisfies the initial conditions $x(0) = 1/10$ and $x'(0) = -1$. A graph of the function is shown in Figure 6.45. Due to excessive damping, oscillations essentially disappear after 3 seconds, but the impulse force restores them at $t = 3$ seconds. Notice the abrupt change in slope (velocity) at $t = 3$ due to the impulse force. Damping again brings the mass essentially to rest after a few seconds.•

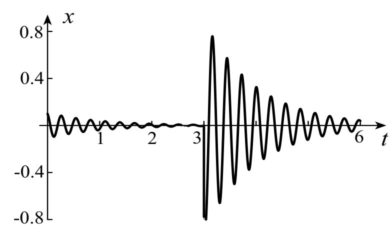


Figure 6.45

We have already seen that when the nonhomogeneity in an n^{th}-order, constant coefficient, linear differential equation is continuous, the solution and its first n derivatives are all continuous. When the nonhomogeneity is piecewise-continuous, the solution and its first $n-1$ derivatives are continuous, but the n^{th} derivative has discontinuities at the discontinuities of the nonhomogeneity. Based upon the above examples, we can expect that when a nonhomogeneity is a Dirac delta function, both the n^{th} and the $(n-1)^{\text{th}}$ derivatives of the solution will have discontinuities.

Transfer Functions and Impulse Response Functions

Consider the initial-value problem consisting of the n^{th}-order constant coefficient differential equation

$$a_n\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0y = f(t), \tag{6.36a}$$

with initial conditions

$$y(0) = y_0, \quad y'(0) = y_0', \quad \ldots \quad , y^{(n-1)}(0) = y_0^{(n-1)}. \tag{6.36b}$$

If we take Laplace transforms, we get

$$a_n[s^nY - s^{n-1}y_0 - s^{n-2}y_0' - \cdots - y_0^{(n-1)}] \\ + a_{n-1}[s^{n-1}Y - s^{n-2}y_0 - \cdots - y_0^{n-2}] + \cdots + a_0Y = F(s).$$

When we solve this for $Y(s)$, we can write the result in the form

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{F(s)}{P(s)}, \tag{6.37}$$

where $P(s) = a_ns^n + a_{n-1}s^{n-1} + \cdots + a_0$, and $Q(s)$ is a polynomial of order less than or equal to $n-1$ determined by the coefficients a_i and the initial conditions.

The thing to notice about this transform is that if $f(t) \equiv 0$, then so also is $F(s)$, and $Y(s) = Q(s)/P(s)$. The inverse transform of this $Y(s)$ can be thought of in two ways. First, if we regard the n initial values as unspecified, then the inverse is a general solution $y_h(t)$ of the homogeneous initial-value problem associated with 6.36. The initial values are the arbitrary constants in the solution. Alternatively, if the initial values are regarded as specified constants, then the inverse of $Y(s)$ is the solution of the homogeneous equation subject to these conditions. Suppose instead that all initial conditions are equal to zero. Then $Q(s) \equiv 0$, and $Y(s) = F(s)/P(s)$. The inverse transform of this is a particular solution $y_p(t)$ of differential equation 6.36 that satisfies $y_p(0) = y_p'(0) = \cdots y_p^{(n-1)} = 0$. Expression 6.37 has separated effects of the initial conditions and the nonhomogeneity into separate terms. Let us illustrate these ideas with an example before proceeding with the discussion.

Example 6.35 A 1-kilogram mass is suspended from a spring with constant 65 newtons per metre. It is put into motion by lifting it 10 centimetres above its equilibrium position and giving it velocity 2 metres per second downward. During its motion, it is subject to a damping force equal to twice the velocity of the mass, and a vertical force $3\sin 4t$. Find expression 6.37 for the problem, and inverse transforms of each term.

Solution The initial-value problem for the motion of the mass is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 65x = 3\sin 4t, \qquad x(0) = 1/10, \quad x'(0) = -2.$$

If we take Lapace transforms,

$$\left(s^2X - \frac{s}{10} + 2\right) + 2\left(sX - \frac{1}{10}\right) + 65X = \frac{12}{s^2+16}.$$

When we solve for $X(s)$,

$$X(s) = \frac{\dfrac{s}{10} - \dfrac{9}{5}}{s^2+2s+65} + \frac{12}{(s^2+16)(s^2+2s+65)}.$$

The first term contains the effect of the initial conditions, and the second term contains the transform of the nonhomogeneity. Inverse transforms of these terms are

$$\begin{aligned}
x_h(t) &= \mathcal{L}^{-1}\left\{\frac{s/10 - 9/5}{s^2+2s+65}\right\} = \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{(s+1)-19}{(s+1)^2+64}\right\} \\
&= \frac{1}{10}e^{-t}\mathcal{L}^{-1}\left\{\frac{s-19}{s^2+64}\right\} = \frac{1}{10}e^{-t}\left(\cos 8t - \frac{19}{8}\sin 8t\right), \\
x_p(t) &= \mathcal{L}^{-1}\left\{\frac{12}{(s^2+16)(s^2+2s+65)}\right\} \\
&= \frac{12}{2465}\mathcal{L}^{-1}\left\{\frac{-2s+49}{s^2+16} + \frac{2s-45}{s^2+2s+65}\right\} \\
&= \frac{12}{2465}\left[\left(-2\cos 4t + \frac{49}{4}\sin 4t\right) + \mathcal{L}^{-1}\left\{\frac{2(s+1)-47}{(s+1)^2+64}\right\}\right] \\
&= \frac{12}{2465}\left[\left(-2\cos 4t + \frac{49}{4}\sin 4t\right) + e^{-t}\left(2\cos 8t - \frac{47}{8}\sin 8t\right)\right].
\end{aligned}$$

It is straightforward to check that $x_h(t)$ satisfies the homogeneous differential equation $x'' + 2x' + 65x = 0$ and the initial conditions $x_h(0) = 1/10$ and $x_h'(0) = -2$. Function $x_p(t)$ satisfies the nonhomogeneous equation $x'' + 2x' + 65x = 3\sin 4t$, and the initial conditions $x_p(0) = x_p'(0) = 0$.•

We now return to discussion of expression 6.37. If we write differential equation 6.36a in operator notation $\phi(D)y = f(t)$, then the function $P(s)$ in expression 6.37 is $\phi(s)$. The function $\mathcal{H}(s) = 1/P(s)$ is called the **transfer function** for system 6.36. Its inverse transform $\hbar(t)$ is called the **unit impulse response function**, because it describes the solution of the initial-value problem when all initial conditions are zero and the nonhomogeneity is the delta function $\delta(t)$. This is easily seen by noting that when all initial conditions are zero and $f(t) = \delta(t)$, then in expression 6.37, $Q(s) = 0$, $F(s) = 1$, and therefore

$$Y(s) = \frac{1}{P(s)} = \mathcal{H}(s).$$

Consequently, $y(t) = \hbar(t)$. We can write the solution of initial-value problem 6.36 in terms of the unit impulse response function. We have already noted that the inverse transform of the first term $Q(s)/P(s)$ in expression 6.37 is $y_h(t)$, the general solution of the associated homogeneous problem. The inverse of the second term $F(s)/P(s)$ can be written as the convolution of $f(t)$ and $\hbar(t)$,

$$\mathcal{L}^{-1}\{F(s)/P(s)\} = \mathcal{L}^{-1}\{F(s)\mathcal{H}(s)\} = \int_0^t f(u)\hbar(t-u)\,du.$$

Thus, the solution of initial-value problem 6.36 is

$$y(t) = y_h(t) + \int_0^t f(u)\hbar(t-u)\,du. \tag{6.38}$$

The first term contains contributions to the solution of the initial-value probem due to the initial conditions, and the second term is the contribution due the nonhomogeneity in the differential equation.

Example 6.36 Use formula 6.38 to find the solution to the initial-value problem in Example 6.35.

Solution Since the auxiliary equation $m^2+2m+65 = 0$ has solutions $m = -1\pm 8i$, a general solution of the associated homogeneous equation is

$$x_h(t) = e^{-t}(C_1\cos 8t + C_2\sin 8t).$$

The initial conditions require $1/10 = x(0) = C_1$, and $-2 = x'(0) = -C_1 + 8C_2$, and therefore the solution of the associated homogeneous differential equation that satisfies the initial conditions is

$$e^{-t}\left(\frac{1}{10}\cos 8t - \frac{19}{80}\sin 8t\right) = \frac{1}{80}e^{-t}(8\cos 8t - 19\sin 8t).$$

The unit impulse response function is

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+65}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+64}\right\} = \frac{1}{8}e^{-t}\sin 8t.$$

If we denote the second term in formula 6.38 by $x_p(t)$, then

$$x_p(t) = \int_0^t 3\sin 4u \left[\frac{1}{8}e^{-(t-u)}\sin 8(t-u)\right] du.$$

Lengthy integration leads to

$$x_p(t) = \frac{147}{2465}\sin 4t - \frac{24}{2465}\cos 4t + \frac{24}{2465}e^{-t}\cos 8t - \frac{141}{4930}e^{-t}\sin 8t.$$

Thus, the solution of the initial-value problem is

$$x(t) = \frac{1}{80}e^{-t}(8\cos 8t - 19\sin 8t) + \frac{147}{2465}\sin 4t - \frac{24}{2465}\cos 4t + \frac{24}{2465}e^{-t}\cos 8t - \frac{141}{4930}e^{-t}\sin 8t. \bullet$$

EXERCISES 6.5

1. A tank contains 1000 litres of water in which 5 kilograms of salt have been dissolved. Starting at time $t = 0$, pure water is added to the tank at 5 millilitres per second. At the same time, mixture starts being drawn from the bottom of the tank at 5 millilitres per second. After 1 minute, 500 grams of salt are suddenly added to the tank, and an additional 500 grams every minute thereafter. Assume that the mixture in the tank is always sufficiently well-stirred that at any given time, concentration of salt is the same at all points in the tank, even when the 500 grams are added to the tank. Find the amount of salt in the tank as a function of time.

2. Repeat Exercise 1 if a brime mixture containing 2 kilograms of salt per 100 litres of water is added to the tank instead of pure water.

3. A 2-kilogram mass is suspended from a spring with constant 512 newtons per metre. If it is set into motion at time $t = 0$ by a unit impulse force, find its subsequent displacement. Assume negligble damping.

4. Repeat Exercise 3 if damping with constant $\beta = 80$ is taken into account.

5. Repeat Exercise 4 if $\beta = 8$.

6. A 2-kilogram mass is suspended from a spring with constant 512 newtons per metre. It is set into motion by moving it to position x_0 and then releasing it. If a unit impulse force is applied at $t_0 > 0$, find the position of the mass for all time.

7. Repeat Exercise 6 if motion is initiated by giving the mass velocity v_0 at time $t = 0$ and position $x = 0$.

8. Repeat Exercise 6 if motion is initiated by giving the mass velocity v_0 from position x_0 at time $t = 0$.

9. A 1-kilogram mass is suspended from a spring with constant 100 newtons per metre. It is subjected to a unit impulse force at $t = 0$ and again at $t = 1$. Find the position of the mass as a function of time.

10. A mass of M kilograms hangs at equilibrium on the end of a spring with constant k newtons per metre. At time $t = 0$, the mass is subjected to a unit impulse force. Show that if the initial-value problem

$$M\frac{d^2x}{dt^2} + kx = \delta(t), \quad x(0) = 0, \quad x'(0) = 0,$$

is solved, the function does not satisfy the initial velocity condition. Can you explain why?

11. Repeat Exercise 9 if unit impulse forces are applied one each second beginning at time $t = 0$. Express the solution in sigma notation.

12. Repeat Exercise 11 if unit impulse forces are $\pi/5$ seconds apart, the first at time $t = 0$. Is there resonance?

13. A mass M, suspended from a spring with constant k, is set into undamped motion by giving it displacement x_0 from its equilibrium position and velocity v_0. At time t_0, it is struck with an impulse force F. Find an expression for F if the impulse force brings the mass to an instantaneous stop.

14. An unstretched spring (with constant k), in the horizontal position, is attached on the left to a wall and on the right to a mass M. At time $t = 0$, the mass is struck with an impulse force F to the right which causes the mass to move. The coefficient of kinetic friction between the mass and the table along which it slides is μ. Assume that damping can be ignored.
(a) What is the initial velocity of the mass as a result of the hit?
(b) When does the mass come to a stop for the first time?
(c) If the coefficient of static friction between the mass and the table is μ_s, show that the mass moves to the left if

$$F > \sqrt{\frac{g^2\mu_s M^3(\mu_s + 2\mu)}{k}}.$$

15. Show that the amplitude of displacement function 6.33b is

$$\frac{2F}{ka}\left|\sin\sqrt{\frac{k}{M}}\frac{a}{2}\right|.$$

Convince yourself that for certain values of a, this could be greater than $F/(ka)$ and for other values of a, it could be less than $F/(ka)$.

16. (a) Suppose the equation of the speed bump in Exercise 19 of Section 5.2 is $f(x) = 3x(1-x)/5$. Find displacement of the front end of the car when $M = 200$ and $k = 1000$. Assume $\beta = 0$ so that shock absorbers are not working, and that the car has slowed down to 20 kilometres per hour.
(b) Show that motion of the car after the speed bump is simple harmonic and find its amplitude.

In Exercises 17–18 find the transfer and unit impulse response functions for the initial-value problem. Express the solution in form 6.38.

17. $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} - 4y = f(t), \quad y(0) = 1, \quad y'(0) = -2$

18. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 3y = f(t), \quad y(0) = A, \quad y'(0) = B$

19. A system modelled by differential equation 6.36a is said to be **stable** if its unit impulse response function $\hbar(t)$ is bounded as $t \to \infty$. Show that this is the case if, and only if, real roots of $P(s) = 0$ are less than or equal to zero, as are real parts of complex roots.

20. A system modelled by differential equation 6.36a is said to be **asymptotically stable** if its unit impulse response function $\hbar(t)$ approaches zero as $t \to \infty$. Show that this is the case if, and only if, real roots of $P(s) = 0$ are negative, as are real parts of complex roots.

Appendix

In the theory of generalized functions, or distributions, functions such as the delta function $\delta(t-t_0)$ are never stand-alone functions; that is, they are not assigned values at specific values of t. It is easy to see why the delta function $\delta(t-t_0)$ should not be considered in a pointwise sense. If it has value zero for $t \neq t_0$, and value "infinity" at t_0, then what is the difference between $\delta(t-t_0)$ and $4\delta(t-t_0)$? Generalized functions are considered to be mappings, or transformations, that map ordinary functions to numbers. For instance, suppose that $g(t)$ is a fixed function that is continuous on the interval $a \le t \le b$. If $f(t)$ is any other function that is integrable on $a \le t \le b$, then the integral

$$\int_a^b f(t)g(t)\,dt$$

is a real number. We can say that through this integration, $g(t)$ defines a mapping from the set of integrable functions on $a \le t \le b$ to the reals. If we denote this mapping by $\mathcal{G}$, then we can write that

$$f(t) \xrightarrow{g(t)} \mathcal{G}\{f\} = \int_a^b g(t)f(t)\,dt. \tag{6.39}$$

For instance, if $g(t) = t^2$, and the interval is $0 \le t \le 2$, then

$$t \xrightarrow{t^2} \mathcal{G}\{t\} = \int_0^2 t(t^2)\,dt = \left\{\frac{t^4}{4}\right\}_0^2 = 4,$$
$$e^t \xrightarrow{t^2} \mathcal{G}\{e^t\} = \int_0^2 t^2e^t dt = \left\{t^2e^t - 2te^t + 2e^t\right\}_0^2 = 2(e^2-1).$$

It is this view of an ordinary function as a mapping, or operator, that is adopted to define $\delta(t-t_0)$. The "generalized" function $\delta(t-t_0)$ is the operator that maps a function $f(t)$, continuous at $t = t_0$, onto its value at $t = t_0$,

$$f(t) \xrightarrow{\delta(t-t_0)} f(t_0). \tag{6.40}$$

For example,

$$t^2 + 2t - 3 \xrightarrow{\delta(t-2)} 5, \qquad \text{and} \qquad (t+1)^2\cos t \xrightarrow{\delta(t)} 1.$$

In order that the delta function have an integral representation, we write

$$f(t) \xrightarrow{\delta(t-t_0)} f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt. \tag{6.41}$$

Because $\delta(t-t_0)$ cannot be regarded pointwise, the multiplication in this integral, and the integral itself, are symbolic. When we encounter an integral such as that in equation 6.41, we interpret it as the action of the function $\delta(t-t_0)$ operating on $f(t)$ and immediately write $f(t_0)$. For example,

$$\int_{-\infty}^{\infty} \left(t^2 + \frac{2}{t^2+1}\right)\delta(t)\,dt = 2, \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t+2)\,dt = 1$$

(since the left side of the latter integral is interpreted as the delta function $\delta(t+2)$ operating on the function $f(t) \equiv 1$).

Because $\delta(t-t_0)$ picks out the value of a function at $t = t_0$, we also write

$$\int_a^b f(t)\delta(t-t_0)\,dt = f(t_0) \tag{6.42a}$$

whenever $a < t_0 < b$; that is, the limits on the integral need not be $\pm\infty$. Furthermore, if $t = t_0$ is not between a and b, we set

$$\int_a^b f(t)\delta(t-t_0)\,dt = 0. \tag{6.42b}$$

For instance,

$$\int_{-2}^{6} \sqrt{t+5}\,\delta(t)\,dt = \sqrt{5}, \qquad \text{and} \qquad \int_2^3 (t^2+2t-4)\delta(t+1)\,dt = 0.$$

With this interpretation for the delta function, its Laplace transform is

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-t_0 s}.$$

What is important to remember from this discussion is that $\delta(t-t_0)$ should never be regarded in a pointwise sense. We see it in initial-value problems such as

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3y = \delta(t-3), \quad y(0) = 1, \quad y'(0) = -2,$$

but we never consider the differential equation at a specific value of t. The first step is always to take the Laplace transform of both sides of the equation, in which case the operational property of the delta function is invoked,

$$[s^2Y - s + 2] + 2[sY - 1] + 3Y = \mathcal{L}\{\delta(t-3)\} = e^{-3s}.$$