CHAPTER 5

EXERCISES 5.1

1. To express the solution in the form $A\cos(40t-\phi)$, we set

$$\frac{1}{10}\cos 40t - \frac{1}{20}\sin 40t = A\cos(40t-\phi) = A[\cos 40t\,\cos\phi + \sin 40t\,\sin\phi].$$

Because $\sin 40t$ and $\cos 40t$ are linearly independent functions we equate coefficients to obtain

$$\frac{1}{10} = A\cos\phi, \qquad \frac{-1}{20} = A\sin\phi.$$

When these are squared and added,

$$\frac{1}{100} + \frac{1}{400} = A^2 \qquad \Longrightarrow \qquad A = \frac{\sqrt{5}}{20}.$$

It now follows that ϕ must satisfy the equations

$$\frac{1}{10} = \frac{\sqrt{5}}{20}\cos\phi, \qquad \frac{-1}{20} = \frac{\sqrt{5}}{20}\sin\phi.$$

One angle satisfying these is $\phi = -0.464$ radians. The position function of the mass can therefore be expressed in the form $\dfrac{\sqrt{5}}{20}\cos(40t+0.464)$.

2. With the coordinate system of Figure 5.6, the initial-value problem describing the position $x(t)$ of the mass is

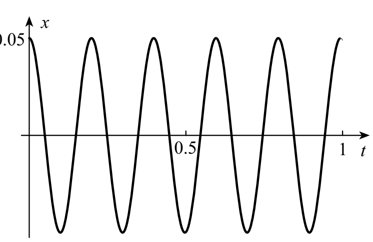
$$(1)\frac{d^2x}{dt^2} + 16x = 0, \qquad x(0) = -1/10, \quad x'(0) = 0.$$

The auxiliary equation is $m^2+16=0$ with solutions $m=\pm 4i$. A general solution of the differential equation is $x(t) = C_1\cos 4t + C_2\sin 4t$. To satisfy the initial conditions, we must have $-1/10 = C_1$ and $0 = 4C_2$. Thus, $x(t) = -(1/10)\cos 4t$ m.

3. With the coordinate system of Figure 5.3, the initial-value problem describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 100x = 0, \qquad x(0) = \frac{1}{20}, \quad x'(0) = 0.$$

The auxiliary equation is $m^2+1000=0$ with solutions $m = \pm 10\sqrt{10}i$. A general solution of the differential equation is $x(t) = C_1\cos 10\sqrt{10}t + C_2\sin 10\sqrt{10}t$. To satisfy the initial conditions, we must have $1/20 = C_1$ and $0 = 10\sqrt{10}C_2$. Thus, $x(t) = (1/20)\cos 10\sqrt{10}t$ m. A graph of this function is shown to the right. The amplitude of the oscillations is 5 cm, the period is $2\pi/(10\sqrt{10}) = \sqrt{10}\pi/50$ s, and the frequency is $50/(\sqrt{10}\pi) = 5\sqrt{10}/\pi$ Hz.

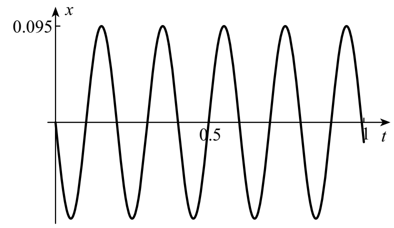


4. With the coordinate system of Figure 5.3, the initial-value problem describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 100x = 0, \qquad x(0) = 0, \quad x'(0) = -3.$$

The auxiliary equation is $m^2+1000=0$ with solutions $m = \pm 10\sqrt{10}i$. A general solution of the differential equation is $x(t) = C_1\cos 10\sqrt{10}t + C_2\sin 10\sqrt{10}t$. To satisfy the initial conditions, we must have $0 = C_1$ and $-3 = 10\sqrt{10}C_2$. Thus,

$x(t) = (-3\sqrt{10}/100)\sin 10\sqrt{10}t$ m. A graph of this function is shown to the right. The amplitude of the oscillations is $3\sqrt{10}/100$ m, the period is $2\pi/(10\sqrt{10}) = \sqrt{10}\pi/50$ s, and the frequency is $50/(\sqrt{10}\pi) = 5\sqrt{10}/\pi$ Hz.



5. With the coordinate system of Figure 5.3, the initial-value problem describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 100x = 0, \qquad x(0) = \frac{1}{20}, \quad x'(0) = -3.$$

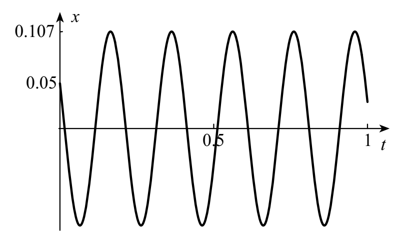
The auxiliary equation is $m^2 + 1000 = 0$ with solutions $m = \pm 10\sqrt{10}i$. A general solution of the differential equation is $x(t) = C_1 \cos 10\sqrt{10}t + C_2 \sin 10\sqrt{10}t$. To satisfy the initial conditions, we must have $1/20 = C_1$ and $-3 = 10\sqrt{10}C_2$. Thus,

$$x(t) = \frac{1}{20}\cos 10\sqrt{10}t - \frac{3\sqrt{10}}{100}\sin 10\sqrt{10}t \text{ m.}$$

A graph of this function is shown to the right. The amplitude of the oscillations is

$$\sqrt{\left(\frac{1}{20}\right)^2 + \left(\frac{-3\sqrt{10}}{100}\right)^2} = \frac{\sqrt{115}}{100} \text{ m.}$$

The period is $2\pi/(10\sqrt{10}) = \sqrt{10}\pi/50$ s, and the frequency is $50/(\sqrt{10}\pi) = 5\sqrt{10}/\pi$ Hz.



6. With the coordinate system of Figure 5.3, the initial-value problem describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 100x = 0, \qquad x(0) = -\frac{1}{20}, \quad x'(0) = -3.$$

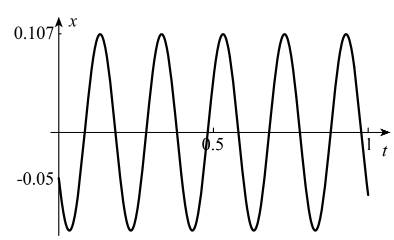
The auxiliary equation is $m^2 + 1000 = 0$ with solutions $m = \pm 10\sqrt{10}i$. A general solution of the differential equation is $x(t) = C_1 \cos 10\sqrt{10}t + C_2 \sin 10\sqrt{10}t$. To satisfy the initial conditions, we must have $-1/20 = C_1$ and $-3 = 10\sqrt{10}C_2$. Thus,

$$x(t) = -\frac{1}{20}\cos 10\sqrt{10}t - \frac{3\sqrt{10}}{100}\sin 10\sqrt{10}t \text{ m.}$$

A graph of this function is shown to the right. The amplitude of the oscillations is

$$\sqrt{\left(\frac{-1}{20}\right)^2 + \left(\frac{-3\sqrt{10}}{100}\right)^2} = \frac{\sqrt{115}}{100} \text{ m.}$$

The period is $2\pi/(10\sqrt{10}) = \sqrt{10}\pi/50$ s, and the frequency is $50/(\sqrt{10}\pi) = 5\sqrt{10}/\pi$ Hz.



7. (a) With the coordinate system of Figure 5.6, the initial-value problem describing the position $x(t)$ of the mass is

$$2\frac{d^2x}{dt^2} + 1000x = 0, \qquad x(0) = -\frac{3}{100}, \quad x'(0) = -2.$$

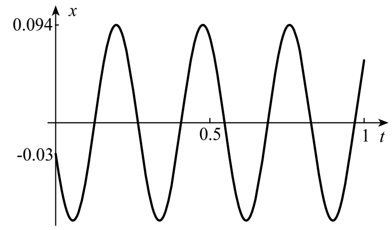
The auxiliary equation is $2m^2 + 1000 = 0$ with solutions $m = \pm 10\sqrt{5}i$. A general solution of the differential equation is $x(t) = C_1 \cos 10\sqrt{5}t + C_2 \sin 10\sqrt{5}t$. To satisfy the initial conditions, we must have $-3/100 = C_1$ and $-2 = 10\sqrt{5}C_2$. Thus,

$$x(t) = -\frac{3}{100}\cos 10\sqrt{5}t - \frac{\sqrt{5}}{25}\sin 10\sqrt{5}t \text{ m.}$$

A graph of this function is shown to the right. The amplitude of the oscillations is

$$\sqrt{\left(\frac{-3}{100}\right)^2 + \left(\frac{-\sqrt{5}}{25}\right)^2} = \frac{\sqrt{89}}{100} \text{ m.}$$

The period is $2\pi/(10\sqrt{5}) = \sqrt{5}\pi/25$ s, and the frequency is $25/(\sqrt{5}\pi) = 5\sqrt{5}/\pi$ Hz.



(b) The initial conditions affect the amplitude, but not the period or frequency.

8. With a mass of 8 kg, the initial-value problem for displacements is

$$8\frac{d^2x}{dt^2} + 1000x = 0, \qquad x(0) = -\frac{3}{100}, \quad x'(0) = -2.$$

The auxiliary equation is $8m^2 + 1000 = 0$ with solutions $m = \pm 5\sqrt{5}i$. A general solution of the differential equation is $x(t) = C_1 \cos 5\sqrt{5}t + C_2 \sin 5\sqrt{5}t$. The period is $2\pi/(5\sqrt{5}) = 2\sqrt{5}\pi/25$ s, double that when the mass was 2 kg. The frequency will be half its previous value.

9. With a spring constant of 4000 N/m, the initial-value problem for displacements is

$$2\frac{d^2x}{dt^2} + 4000x = 0, \qquad x(0) = -\frac{3}{100}, \quad x'(0) = -2.$$

The auxiliary equation is $2m^2 + 4000 = 0$ with solutions $m = \pm 20\sqrt{5}i$. A general solution of the differential equation is $x(t) = C_1 \cos 20\sqrt{5}t + C_2 \sin 20\sqrt{5}t$. The period is $2\pi/(20\sqrt{5}) = \sqrt{5}\pi/50$ s, half that when the spring constant was 1000 N/m. The frequency will be double its previous value.

10. With the coordinate system of Figure 5.6, the differential equation describing the position $x(t)$ of the mass is

$$2\frac{d^2x}{dt^2} + kx = 0.$$

The auxiliary equation is $2m^2 + k = 0$ with solutions $m = \pm\sqrt{k/2}i$. A general solution of the differential equation is $x(t) = C_1 \cos\sqrt{k/2}t + C_2 \sin\sqrt{k/2}t$. The period of the oscillations is $2\pi/\sqrt{k/2}$ and therefore the frequency is $\sqrt{k/2}/(2\pi)$ Hz. Since this must be 3, we set $\sqrt{k/2}/(2\pi) = 3$, from which $k = 72\pi^2$ N/m.

11. With the coordinate system of Figure 5.6, the initial-value problem describing the position $x(t)$ of the mass is

$$M\frac{d^2x}{dt^2} + kx = 0, \qquad x(0) = x_0, \quad x'(0) = v_0.$$

The auxiliary equation is $Mm^2 + k = 0$ with solutions $m = \pm\sqrt{k/M}i$. A general solution of the differential equation is $x(t) = C_1 \cos\sqrt{k/M}t + C_2 \sin\sqrt{k/M}t$. To satisfy the initial conditions, we must have $x_0 = x(0) = C_1$ and $v_0 = x'(0) = \sqrt{k/M}C_2$. Thus,

$$x(t) = x_0 \cos\sqrt{\frac{k}{M}}t + \sqrt{\frac{M}{k}}v_0 \sin\sqrt{\frac{k}{M}}t.$$

If we set this equal to $A\sin(\sqrt{k/M}t + \phi)$, then

$$x_0 \cos\sqrt{\frac{k}{M}}t + \sqrt{\frac{M}{k}}v_0 \sin\sqrt{\frac{k}{M}}t = A\left(\sin\sqrt{\frac{k}{M}}t\cos\phi + \cos\sqrt{\frac{k}{M}}t\sin\phi\right).$$

This implies that

$$x_0 = A\sin\phi, \qquad \sqrt{\frac{M}{k}}v_0 = A\cos\phi.$$

When these are squared and added,

$$A^2 = x_0^2 + \frac{Mv_0^2}{k} \qquad \Longrightarrow \qquad A = \sqrt{x_0^2 + \frac{Mv_0^2}{k}}.$$

It then follows that

$$\sin\phi = \frac{x_0}{A}, \qquad \cos\phi = \frac{\sqrt{M/k}v_0}{A}.$$

12. The period of the oscillations in Exercise 11 is $2\pi/\sqrt{k/M} = 2\pi\sqrt{M/k}$. This formula makes it clear that when M is doubled, the period is increased by a factor of $\sqrt{2}$. It follows that the frequency must be decreased by the same factor.

13. (a) When damping is ignored, the differential equation describing displacements of a mass is

$$M\frac{d^2x}{dt^2} + kx = 0.$$

Since velocity is a maximum when acceleration is zero, it follows that velocity is a maximum when $x = 0$; that is, the mass passes through the equilibrium position.
(b) Maximum acceleration occurs when $d^3x/dt^3 = 0$, and the differential equation implies that this occurs when $dx/dt = 0$; that is, when the velocity of the mass is zero. This occurs when the mass is at its maximum distance from equilibrium.

14. If we use differential equation 5.7 to describe oscillations of the mass, there is no difference in the analysis.

15. (a) With the coordinate system of Figure 5.6, the initial-value problem describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 40x = 0, \qquad x(0) = -\frac{1}{50}, \quad x'(0) = 10.$$

The auxiliary equation is $m^2 + 400 = 0$ with solutions $m = \pm 20i$. A general solution of the differential equation is $x(t) = C_1\cos 20t + C_2\sin 20t$. To satisfy the initial conditions, we must have $-1/50 = C_1$ and $10 = 20C_2$. Thus,

$$x(t) = -\frac{1}{50}\cos 20t + \frac{1}{2}\sin 20t \text{ m}.$$

(b) To simplify the remaining parts of the exercise we express $x(t)$ in the form

$$-\frac{1}{50}\cos 20t + \frac{1}{2}\sin 20t = A\sin(20t+\phi) = A(\sin 20t\cos\phi + \cos 20t\sin\phi).$$

These imply that

$$-\frac{1}{50} = A\sin\phi, \qquad \frac{1}{2} = A\cos\phi.$$

When these are squared and added,

$$A^2 = \left(\frac{-1}{50}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{626}{2500} \qquad \Longrightarrow \qquad A = \frac{\sqrt{626}}{50}.$$

With this value for A,

$$\sin\phi = -\frac{1}{\sqrt{626}}, \qquad \cos\phi = \frac{25}{\sqrt{626}}.$$

One of many expressions for ϕ is $\phi = -\mathrm{Sin}^{-1}(1/\sqrt{626})$. Thus,

$$x(t) = \frac{\sqrt{626}}{50}\sin(20t - \theta), \quad \text{where } \theta = \text{Sin}^{-1}\left(\frac{1}{\sqrt{626}}\right).$$

The amplitude of the motion is $\sqrt{626}/50$ m, the period is $\pi/10$ s and the frequency is $10/\pi$ Hz.
(c) The velocity of the mass is zero when

$$0 = x'(t) = \frac{20\sqrt{626}}{50}\cos(20t - \theta) \quad \Longrightarrow \quad 20t - \theta = \frac{(2n+1)\pi}{2} \quad \Longrightarrow \quad t = \frac{(2n+1)\pi}{40} + \frac{\theta}{20},$$

where $n \geq 0$ is an integer.
(d) The mass passes through the equilibrium point when

$$0 = x(t) \quad \Longrightarrow \quad 20t - \theta = n\pi \quad \Longrightarrow \quad t = \frac{\theta}{20} + \frac{n\pi}{20},$$

where $n \geq 0$ is an integer.
(e) The mass is 1 cm above its equilibrium position when

$$\frac{1}{100} = \frac{\sqrt{626}}{50}\sin(20t - \theta) \quad \Longrightarrow \quad \sin(20t - \theta) = \frac{1}{2\sqrt{626}}.$$

This is true when

$$20t - \theta = \begin{cases} \text{Sin}^{-1}\left(\dfrac{1}{2\sqrt{626}}\right) + 2n\pi \\ \pi - \text{Sin}^{-1}\left(\dfrac{1}{2\sqrt{626}}\right) + 2n\pi \end{cases} \quad \Longrightarrow \quad t = \begin{cases} \dfrac{1}{20}\text{Sin}^{-1}\left(\dfrac{1}{2\sqrt{626}}\right) + \dfrac{\theta}{20} + \dfrac{n\pi}{10} \\ -\dfrac{1}{20}\text{Sin}^{-1}\left(\dfrac{1}{2\sqrt{626}}\right) + \dfrac{\theta}{20} + \dfrac{(2n+1)\pi}{20}, \end{cases}$$

where $n \geq 0$ is an integer.
(f) The velocity of the mass is 12 if, and when,

$$12 = \frac{2\sqrt{626}}{5}\cos(20t - \theta) \quad \Longrightarrow \quad \cos(20t - \theta) = \frac{30}{\sqrt{626}} > 1.$$

Hence, the mass never attains this velocity.
(g) The mass is at maximum height when

$$\frac{\sqrt{626}}{50} = \frac{\sqrt{626}}{50}\sin(20t - \theta) \quad \Longrightarrow \quad \sin(20t - \theta) = 1 \quad \Longrightarrow \quad 20t - \theta = \frac{(4n+1)\pi}{2}$$
$$\Longrightarrow \quad t = \frac{\theta}{20} + \frac{(4n+1)\pi}{40},$$

where $n \geq 0$ is an integer. This happens for the second time when $n = 1$, in which case $t = \theta/20 + \pi/8$.

16. (a) When we write expression 5.9 in the form $A\sin(\omega t - \phi)$,

$$C_1\cos\omega t + C_2\sin\omega t = A\sin(\omega t - \phi) = A(\sin\omega t\cos\phi - \cos\omega t\sin\phi).$$

When we equate coefficients of $\cos\omega t$ and $\sin\omega t$, we obtain

$$C_1 = -A\sin\phi, \quad C_2 = A\cos\phi \quad \Longrightarrow \quad \sin\phi = -\frac{C_1}{A}, \quad \cos\phi = \frac{C_2}{A}.$$

Similar derivations give the equations in part (b) and (c).

17. If s is the stretch in the spring at equilibrium, then $ks = Mg$ so that $s = Mg/k$. This is the initial displacement of the mass relative to the equilibrium position. The initial-value problem describing the position $x(t)$ of the mass relative to the equilibrium position is

$$M\frac{d^2x}{dt^2} + kx = 0, \qquad x(0) = \frac{Mg}{k}, \quad x'(0) = 0.$$

The auxiliary equation is $Mm^2 + k = 0$ with solutions $m = \pm\sqrt{k/M}i$. A general solution of the differential equation is $x(t) = C_1 \cos(\sqrt{k/M}t) + C_2 \sin(\sqrt{k/M}t)$. To satisfy the initial conditions, we must have $Mg/k = C_1$ and $0 = \sqrt{k/M}C_2$. Thus,

$$x(t) = \frac{Mg}{k} \cos\sqrt{\frac{k}{M}}t \text{ m.}$$

18. According to equation 5.4, the differential equation for displacement of the mass is

$$M\frac{d^2x}{dt^2} + kx = kA\sin\omega t,$$

subject to the initial conditions $x(0) = x_0$ and $x'(0) = v_0$. Since roots of the auxiliary equation $Mm^2 + k = 0$ are $m = \pm\sqrt{k/M}i$, a general solution of the associated homogeneous equation is

$$x_h(t) = C_1 \cos\sqrt{\frac{k}{M}}t + C_2 \sin\sqrt{\frac{k}{M}}t.$$

Assuming a particular solution of the form $x_p(t) = B\sin\omega t + D\cos\omega t$, and substituting into the differential equation,

$$M(-\omega^2 B\sin\omega t - \omega^2 D\cos\omega t) + k(B\sin\omega t + D\cos\omega t) = kA\sin\omega t.$$

Equating coefficients of $\sin\omega t$ and $\cos\omega t$ gives

$$-M\omega^2 B + kB = kA, \qquad -M\omega^2 D + kD = 0.$$

Thus, $D = 0$ and $B = kA/(k - M\omega^2)$, and

$$x(t) = C_1 \cos\sqrt{\frac{k}{M}}t + C_2 \sin\sqrt{\frac{k}{M}}t + \frac{kA}{k - M\omega^2}\sin\omega t.$$

The initial conditions require

$$x_0 = C_1, \quad v_0 = \sqrt{\frac{k}{M}}C_2 + \frac{kA\omega}{k - M\omega^2}.$$

The second of these gives $C_2 = \sqrt{\frac{M}{k}}\left(v_0 - \frac{kA}{k - M\omega^2}\right)$, and

$$x(t) = x_0 \cos\sqrt{\frac{k}{M}}t + \sqrt{\frac{M}{k}}\left(v_0 - \frac{kA}{k - M\omega^2}\right)\sin\sqrt{\frac{k}{M}}t + \frac{kA}{k - M\omega^2}\sin\omega t \text{ m.}$$

19. When the surface of the liquid in the right half of the tube is y metres above the equilibrium position, the mass of liquid in the right tube above the surface in the left tube is $2\pi r^2\rho y$. The force of gravity on this much of the liquid acts on all of the liquid in the tube. In other words, if M represents the mass of liquid in the tube, Newton's second law for motion of the surface in the right half of the tube is

$$M\frac{d^2y}{dt^2} = -2\pi r^2\rho g y.$$

Notice that this equation is valid even when the level of liquid in the right part of the tube is below its equilibrium position. Since the auxiliary equation is $Mm^2 + 2\pi r^2\rho g = 0$, with roots $m = \pm\sqrt{2\pi r^2\rho g/M}$, displacement of the right surface is of the form

$$y(t) = C_1 \cos\sqrt{\frac{2\pi r^2\rho g}{M}}t + C_2 \sin\sqrt{\frac{2\pi r^2\rho g}{M}}t.$$

This is simple harmonic motion with period $\dfrac{2\pi}{\sqrt{2\pi r^2\rho g/M}} = \dfrac{1}{r}\sqrt{\dfrac{2\pi M}{\rho g}}$ s.

20. From time $t = 0$ when the container is attached to the spring until water has completely drained out, the mass of the container is $M - rt$.

(a) With the coordinate system of Figure 5.5, Newton's second law 3.4 gives

$$\frac{d}{dt}\left[(M - rt)\frac{dy}{dt}\right] = -(M - rt)g - \beta\frac{dy}{dt} - ky.$$

By expanding the first term, we can write the differential equation in the form

$$(M - rt)\frac{d^2y}{dt^2} + (\beta - r)\frac{dy}{dt} + ky = -(M - rt)g.$$

The initial-value problem is this differential equation subject to $y(0) = 0 = y'(0)$.

(b) Consider now the coordinate system of Figure 5.6 where $x = 0$ corresponds to the position of the container were it full and at equilibrium. The stretch s in the spring at this position is given by the equation $ks - Mg = 0$. Newton's second law gives

$$\frac{d}{dt}\left[(M - rt)\frac{dx}{dt}\right] = -(M - rt)g - \beta\frac{dx}{dt} + k(s - x).$$

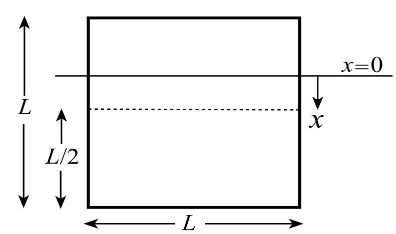
When we expand and use the equation $Mg - ks = 0$, we find

$$(M - rt)\frac{d^2x}{dt^2} + (\beta - r)\frac{dx}{dt} + kx = rgt.$$

The initial-value problem is this differential equation subject to $x(0) = Mg/k$ and $x'(0) = 0$. Because the coefficient of the second derivative in both equations is not constant, we cannot solve the differential equation with the techniques that we now have available.

21. (a) Since the cube floats half submerged, its density is one-half that of water, namely 500 kg/m^3. Suppose we let x denote the distance of the midpoint of the cube below the surface of the water. When the midpoint is x m below the surface, the force on the cube is the buoyant force due to Archimedes' principle less the force of gravity,

$$-9810L^2\left(\frac{L}{2} + x\right) + 4905L^3 = -9810L^2x.$$



The differential equation describing oscillations of the cube is therefore

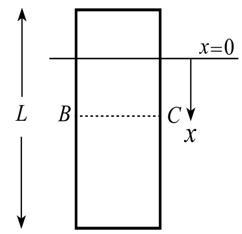
$$500L^3\frac{d^2x}{dt^2} = -9810L^2x \quad\Longrightarrow\quad x'' + \frac{981}{50L}x = 0.$$

(b) The auxiliary equation $m^2 + 981/(50L) = 0$ has solutions $m = \pm\sqrt{981/(50L)}\,i$, and therefore

$$x(t) = C_1\cos\sqrt{\frac{981}{50L}}t + C_2\sin\sqrt{\frac{981}{50L}}t.$$

The frequency of the oscillations is $\dfrac{\sqrt{981/(50L)}}{2\pi} = \dfrac{0.705}{\sqrt{L}}$ Hz.

22. Let BC be the line on the cylinder that resides in the surface of the water when the cylinder is at equilibrium. If x represents the depth of BC below the surface when the cylinder is in motion, then Newton's second law for the acceleration of the cylinder is



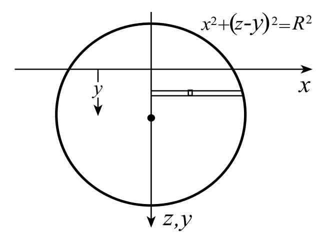
$$M\frac{d^2x}{dt^2} = -9.81(1000)(Ax),$$

where M is the mass of the cylinder and A is its cross-sectional area. Since $M = \rho AL$, where L is the length of the cylinder, and ρ is its density

$$\rho AL\frac{d^2x}{dt^2} = -9810Ax \implies L\rho\frac{d^2x}{dt^2} + 9810x = 0.$$

The auxiliary equation $L\rho m^2 + 9810 = 0$ has roots $m = \pm\sqrt{9810/(L\rho)}i$, so that $x(t) = C_1\cos\sqrt{9810/(L\rho)}t + C_2\sin\sqrt{9810/(L\rho)}t$. Since the period of the oscillations is 4 s, it follows that $2\pi\sqrt{L\rho/9810} = 4 \implies L = 39\,240/\rho\pi^2$. The mass of the cylinder is therefore $\rho AL = \rho(\pi/100)(39\,240/(\rho\pi^2)) = 124.9$ kg.

23. Because the sphere floats half submerged, its density is one-half that of water, namely 500 kg/m^3. The resultant vertical force on the sphere when its centre is y m below the surface is the buoyant force due to the water displaced by the sphere less the force of gravity on the sphere,



$$-9810V + 4905\left(\frac{4}{3}\right)\pi R^3,$$

where V is the volume of water displaced by the sphere when its centre is y m below the surface. We can calculate V with the following double iterated integral,

$$\begin{aligned} V &= \int_0^{R+y}\int_0^{\sqrt{R^2-(z-y)^2}} 2\pi x\,dx\,dz = 2\pi\int_0^{R+y}\left\{\frac{x^2}{2}\right\}_0^{\sqrt{R^2-(z-y)^2}} dz \\ &= \pi\int_0^{R+y}[R^2-(z-y)^2]\,dz = \pi\left\{R^2z - \frac{(z-y)^3}{3}\right\}_0^{R+y} = \frac{\pi}{3}(2R^3+3R^2y-y^3). \end{aligned}$$

The resultant force on the sphere when its centre is at depth y is therefore

$$\frac{-9810\pi}{3}(2R^3+3R^2y-y^3) + \frac{19\,620}{3}\pi R^3 = \frac{9810\pi}{3}(y^3-3R^2y).$$

Newton's second law now gives

$$\frac{4}{3}\pi R^3(500)\frac{d^2y}{dt^2} = \frac{9810\pi}{3}(y^3-3R^2y) \quad\implies\quad \frac{d^2y}{dt^2} = -\frac{3(9.81)}{2R^3}\left(R^2y - \frac{y^3}{3}\right).$$

This is not a linear equation.

EXERCISES 5.2

1. The initial-value problem describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2}+\frac{1}{10}\frac{dx}{dt}+16x=0, \qquad x(0)=-\frac{1}{10}, \quad x'(0)=0.$$

The auxiliary equation is $10m^2+m+160=0$ with solutions $m=(-1\pm 9\sqrt{79}i)/20$. A general solution of the differential equation is $x(t)=e^{-t/20}[C_1\cos(9\sqrt{79}t/20)+C_2\sin(9\sqrt{79}t/20)]$. To satisfy the initial conditions, we must have $-1/10=C_1$ and $0=-C_1/20+9\sqrt{79}C_2/20$. These give

$$x(t)=e^{-t/20}\left(-\frac{1}{10}\cos\frac{9\sqrt{79}t}{20}-\frac{\sqrt{79}}{7110}\sin\frac{9\sqrt{79}t}{20}\right) \text{ m.}$$

2. The initial-value problem describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2}+10\frac{dx}{dt}+16x=0, \qquad x(0)=-\frac{1}{10}, \quad x'(0)=0.$$

The auxiliary equation is $m^2+10m+16=0$ with solutions $m=-2,\,-8$. A general solution of the differential equation is $x(t)=C_1e^{-2t}+C_2e^{-8t}$. The initial conditions require $-1/10=C_1+C_2$ and $0=-2C_1-8C_2$. These give $C_1=-2/15$ and $C_2=1/30$. Thus, $x(t)=(e^{-8t}-4e^{-2t})/30$ m.

3. The differential equation for motion with an unspecified damping factor is

$$(1)\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+16x=0.$$

Critically damped motion ocurs when roots of the auxiliary equation $m^2+\beta m+16=0$ are real and equal, and this occurs when the discriminant of the quadratic is equal to zero,

$$\beta^2-4(1)(16)=0 \qquad \Longrightarrow \qquad \beta=8.$$

4. The initial-value problem describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2}+40\frac{dx}{dt}+4000x=0, \qquad x(0)=\frac{1}{50}, \quad x'(0)=-4.$$

The auxiliary equation is $m^2+400m+40\,000=0$ with solutions $m=-200,-200$. A general solution of the differential equation is $x(t)=(C_1+C_2t)e^{-200t}$. To satisfy the initial conditions, we must have

$$\frac{1}{50}=C_1, \quad -4=-200C_1+C_2 \qquad \Longrightarrow \qquad C_2=0.$$

Thus, $x(t)=(1/50)e^{-200t}$ m. Since this function is never equal to zero, the mass does not pass through the equilibrium position.

5. The initial-value problem describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2}+40\frac{dx}{dt}+4000x=0, \qquad x(0)=\frac{1}{50}, \quad x'(0)=-10.$$

The auxiliary equation is $m^2+400m+40\,000=0$ with solutions $m=-200,-200$. A general solution of the differential equation is $x(t)=(C_1+C_2t)e^{-200t}$. To satisfy the initial conditions, we must have

$$\frac{1}{50}=C_1, \quad -10=-200C_1+C_2 \qquad \Longrightarrow \qquad C_2=-6.$$

Thus, $x(t)=\left(\dfrac{1}{50}-6t\right)e^{-200t}$ m. The mass passes through the equilibrium position if this function is ever equal to zero,

$$\left(\frac{1}{50} - 6t\right) e^{-200t} = 0 \qquad \Longrightarrow \qquad t = \frac{1}{300} \text{ s.}$$

6. (a) The initial-value problem describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2} + 15\frac{dx}{dt} + 50x = 0, \qquad x(0) = \frac{1}{20}, \quad x'(0) = 3.$$

The auxiliary equation is $m^2 + 15m + 50 = 0$ with solutions $m = -5, -10$. A general solution of the differential equation is $x(t) = C_1e^{-5t} + C_2e^{-10t}$. To satisfy the initial conditions, we must have

$$\frac{1}{20} = C_1 + C_2, \quad 3 = -5C_1 - 10C_2 \qquad \Longrightarrow \qquad C_1 = \frac{7}{10}, \quad C_2 = -\frac{13}{20}.$$

Thus, $x(t) = \frac{1}{20}\left(14e^{-5t} - 13e^{-10t}\right)$ m.

(b) The mass passes through the equilibrium position if this function is ever equal to zero,

$$\frac{1}{20}\left(14e^{-5t} - 13e^{-10t}\right) = 0 \qquad \Longrightarrow \qquad e^{5t} = \frac{13}{14}.$$

Since this cannot happen for $t > 0$, the mass does not pass through its equilibrium position.

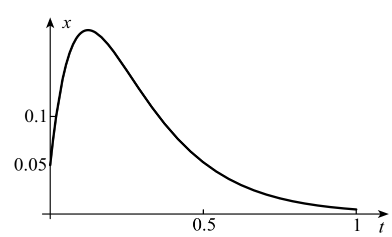
(c) The mass is 1 cm above the equilibrium position when

$$\frac{1}{20}\left(14e^{-5t} - 13e^{-10t}\right) = \frac{1}{100} \qquad \Longrightarrow e^{10t} - 70e^{5t} + 65 = 0.$$

Solutions of this quadratic equation in e^{5t} are

$$e^{5t} = \frac{70 \pm \sqrt{4900 - 4(65)}}{2} = 35 \pm 2\sqrt{290}.$$

Since t must be positive, we take the positive root, in which case $t = (1/5)\ln\left(35 + 2\sqrt{290}\right)$ s.



7. (a) The initial-value problem describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2} + 15\frac{dx}{dt} + 50x = 0, \qquad x(0) = \frac{1}{20}, \quad x'(0) = -\frac{3}{4}.$$

The auxiliary equation is $m^2 + 15m + 50 = 0$ with solutions $m = -5, -10$. A general solution of the differential equation is $x(t) = C_1e^{-5t} + C_2e^{-10t}$. To satisfy the initial conditions, we must have

$$\frac{1}{20} = C_1 + C_2, \quad -\frac{3}{4} = -5C_1 - 10C_2 \qquad \Longrightarrow \qquad C_1 = -\frac{1}{20}, \quad C_2 = \frac{1}{10}.$$

Thus, $x(t) = \frac{1}{20}\left(2e^{-10t} - e^{-5t}\right)$ m.

(b) The mass passes through the equilibrium position if this function is ever equal to zero,

$$\frac{1}{20}\left(2e^{-10t} - e^{-5t}\right) = 0 \qquad \Longrightarrow \qquad e^{5t} = 2 \qquad \Longrightarrow \qquad t = \frac{1}{5}\ln 2 \text{ s.}$$

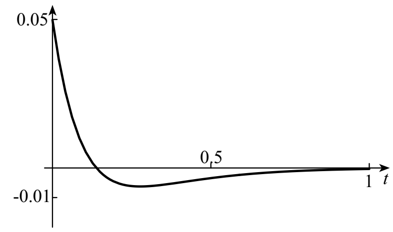
(c) The mass is 1 cm above the equilibrium position when

$$\frac{1}{20}\left(2e^{-10t} - e^{-5t}\right) = \frac{1}{100} \qquad \Longrightarrow e^{10t} + 5e^{5t} - 10 = 0.$$

Solutions of this quadratic equation in e^{5t} are

$$e^{5t} = \frac{-5 \pm \sqrt{25+40}}{2} = \frac{-5 \pm \sqrt{65}}{2}.$$

Since t must be positive, we take the positive root, in which case $t = (1/5)\ln\left(\frac{-5+\sqrt{65}}{2}\right)$ s.



The figure makes it clear that the mass never never reaches 1 cm below the equilibrium position. We can confirm this algebraically by setting

$$\frac{1}{20}(2e^{-10t} - e^{-5t}) = -\frac{1}{100} \qquad \Longrightarrow \qquad e^{10t} - 5e^{5t} + 10 = 0.$$

Solutions of this quadratic are $e^{5t} = \dfrac{5 \pm \sqrt{25-40}}{2}$, which are complex.

8. (a) The initial-value problem describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2} + 15\frac{dx}{dt} + 50x = 0, \qquad x(0) = \frac{1}{20}, \quad x'(0) = -3.$$

The auxiliary equation is $m^2 + 15m + 50 = 0$ with solutions $m = -5, -10$. A general solution of the differential equation is $x(t) = C_1e^{-5t} + C_2e^{-10t}$. To satisfy the initial conditions, we must have

$$\frac{1}{20} = C_1 + C_2, \quad -3 = -5C_1 - 10C_2 \qquad \Longrightarrow \qquad C_1 = -\frac{1}{2}, \quad C_2 = \frac{11}{20}.$$

Thus, $x(t) = \dfrac{1}{20}\left(11e^{-10t} - 10e^{-5t}\right)$ m.

(b) The mass passes through the equilibrium position if this function is ever equal to zero,

$$\frac{1}{20}\left(11e^{-10t} - 10e^{-5t}\right) = 0 \qquad \Longrightarrow \qquad e^{5t} = \frac{11}{10} \qquad \Longrightarrow \qquad t = \frac{1}{5}\ln(11/10) \text{ s}.$$

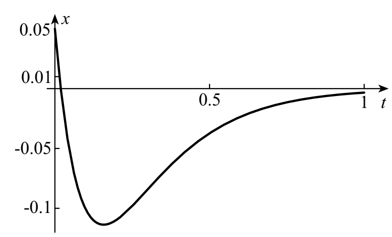
(c) The mass is 1 cm above the equilibrium position when

$$\frac{1}{20}\left(11e^{-10t} - 10e^{-5t}\right) = \frac{1}{100} \qquad \Longrightarrow e^{10t} + 50e^{5t} - 55 = 0.$$

Solutions of this quadratic equation in e^{5t} are

$$e^{5t} = \frac{-50 \pm \sqrt{2500+220}}{2} = -25 \pm 2\sqrt{170}.$$

Since t must be positive, we take the positive root, in which case $t = (1/5)\ln(2\sqrt{170} - 25)$ s.



The mass is 1 cm below the equilibrium position when

$$\frac{1}{20}\left(11e^{-10t} - 10e^{-5t}\right) = -\frac{1}{100} \qquad \Longrightarrow e^{10t} - 50e^{5t} + 55 = 0.$$

Solutions of this quadratic equation in e^{5t} are

$$e^{5t} = \frac{50 \pm \sqrt{2500-220}}{2} = 25 \pm \sqrt{570} \qquad \Longrightarrow \qquad t = \frac{1}{5}\ln(25 \pm \sqrt{570}) \text{ s}.$$

9. (a) The initial-value problem describing the position $x(t)$ of the mass is

$$2\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 200x = 0, \qquad x(0) = \frac{1}{10}, \quad x'(0) = 5.$$

The auxiliary equation is $2m^2+4m+200=0$ with solutions $m=-1\pm 3\sqrt{11}i$. A general solution of the differential equation is $x(t)=e^{-t}[C_1\cos(3\sqrt{11}t)+C_2\sin 3\sqrt{11}t)]$. To satisfy the initial conditions, we must have

$$\frac{1}{10}=C_1,\quad 5=-C_1+3\sqrt{11}C_2 \qquad\Longrightarrow\qquad C_2=\frac{17\sqrt{11}}{110}.$$

Thus, $x(t)=\dfrac{e^{-t}}{110}\left[11\cos(3\sqrt{11}t)+17\sqrt{11}\sin(3\sqrt{11}t)\right]$ m.

(b) Maximum distance from equilibrium is attained when velocity is equal to zero for the first time,

$$0=x'(t)=-\frac{e^{-t}}{110}\left[11\cos(3\sqrt{11}t)+17\sqrt{11}\sin(3\sqrt{11}t)\right]+\frac{e^{-t}}{110}\left[-33\sqrt{11}\sin(3\sqrt{11}t)+561\cos(3\sqrt{11}t)\right].$$

This equation implies that

$$550\cos(3\sqrt{11}t)=50\sqrt{11}\sin(3\sqrt{11}t) \quad\Longrightarrow\quad \tan(3\sqrt{11}t)=\sqrt{11} \quad\Longrightarrow\quad t=\frac{1}{3\sqrt{11}}\mathrm{Tan}^{-1}\sqrt{11}+\frac{n\pi}{3\sqrt{11}},$$

where $n\geq 0$ is an integer. We choose $n=0$ for maximum distance, in which case $t=\dfrac{1}{3\sqrt{11}}\mathrm{Tan}^{-1}\sqrt{11}$. When this is substituted into $x(t)$, the result is $x=0.457$ m or 45.7 cm.

(c) The mass passes through the equilibrium position when

$$0=x(t)=\frac{e^{-t}}{110}\left[11\cos(3\sqrt{11}t)+17\sqrt{11}\sin(3\sqrt{11}t)\right] \quad\Longrightarrow\quad \tan(3\sqrt{11}t)=-\frac{\sqrt{11}}{17}.$$

Thus, $t=\dfrac{1}{3\sqrt{11}}\mathrm{Tan}^{-1}\left(\dfrac{-\sqrt{11}}{17}\right)+\dfrac{n\pi}{3\sqrt{11}}$, where $n\geq 1$ is an integer. When we choose $n=1$ for the first pass through the origin, $t=\dfrac{1}{3\sqrt{11}}\mathrm{Tan}^{-1}\left(\dfrac{-\sqrt{11}}{17}\right)+\dfrac{\pi}{3\sqrt{11}}\approx 0.296$ s.

10. (a) The initial-value problem describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2}+2\frac{dx}{dt}+40x=0, \qquad x(0)=-\frac{1}{20},\quad x'(0)=0.$$

The auxiliary equation is $m^2+2m+40=0$ with solutions $m=-1\pm\sqrt{39}i$. A general solution of the differential equation is $x(t)=e^{-t}(C_1\cos\sqrt{39}t+C_2\sin\sqrt{39}t)$. To satisfy the initial conditions, we must have

$$-\frac{1}{20}=C_1,\quad 0=-C_1+\sqrt{39}C_2 \qquad\Longrightarrow\qquad C_2=-\frac{\sqrt{39}}{780}.$$

Thus, $x(t)=-\dfrac{e^{-t}}{780}\left[39\cos\sqrt{39}t+\sqrt{39}\sin\sqrt{39}t\right]$ m. We now set

$$-\frac{1}{780}(39\cos\sqrt{39}t+\sqrt{39}\sin\sqrt{39}t)=A\sin(\sqrt{39}t+\phi)=A(\sin\sqrt{39}t\cos\phi+\cos\sqrt{39}t\sin\phi).$$

This implies that

$$A\cos\phi=-\frac{\sqrt{39}}{780}, \qquad A\sin\phi=-\frac{39}{780}.$$

Squaring and adding these gives

$$A^2=\frac{39}{780^2}+\frac{39^2}{780^2}=\frac{1}{390} \qquad\Longrightarrow\quad A=\frac{1}{\sqrt{390}}.$$

Hence,

$$\cos\phi = -\frac{\sqrt{39}\sqrt{390}}{780}, \qquad \sin\phi = -\frac{39\sqrt{390}}{780}.$$

One solution of these equations is $\phi = -1.73$. Thus, $x(t) = \dfrac{e^{-t}}{\sqrt{390}}\sin(\sqrt{39}t - 1.73)$.

(b) The mass passes through the equilibrium position when

$$0 = x(t) = \frac{1}{\sqrt{390}}\sin(\sqrt{39}t - 1.73) \quad\Longrightarrow\quad \sqrt{39}t - 1.73 = n\pi \quad\Longrightarrow\quad t = \frac{n\pi + 1.73}{\sqrt{39}},$$

where $n \geq 0$ is an integer. The distance between successive times is $\pi/\sqrt{39}$. The differential equation for the undamped system is

$$\frac{d^2x}{dt^2} + 40x = 0,$$

with auxiliary equation $m^2 + 40 = 0$. Roots are $m = \pm 2\sqrt{[10]}i$, so that the solution of the differential equation is $x(t) = C_1\cos 2\sqrt{10}t + C_2\sin 2\sqrt{10}t$. The period of undamped oscillations is therefore $2\pi/(2\sqrt{10}) = \pi/\sqrt{10}$. This is not the same as $2\pi/\sqrt{39}$.

11. (a) The initial-value problem describing the position $x(t)$ of the mass is

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0, \qquad x(0) = x_0, \quad x'(0) = v_0.$$

The auxiliary equation is $Mm^2 + \beta m + k = 0$ with solutions $m = \dfrac{-\beta \pm \sqrt{\beta^2 - 4kM}}{2M}$. Since the motion is critically damped, $\beta^2 - 4kM = 0$, and the auxiliary has equal roots $m = -\beta/(2M)$. A general solution of the differential equation is $x(t) = (C_1 + C_2 t)e^{-\beta t/(2M)}$. To satisfy the initial conditions, we must have

$$x_0 = C_1, \quad v_0 = C_2 - \frac{\beta}{2M}C_1 \qquad\Longrightarrow\qquad C_2 = v_0 + \frac{\beta x_0}{2M}.$$

Thus, $x(t) = \left[x_0 + \left(v_0 + \dfrac{\beta x_0}{2M}\right)t\right]e^{-\beta t/(2M)}$ m. The mass passes through the equilibrium position when

$$0 = x(t) = \left[x_0 + \left(v_0 + \frac{\beta x_0}{2M}\right)t\right]e^{-\beta t/(2M)} \qquad\Longrightarrow\qquad t = -\frac{x_0}{v_0 + \beta x_0/(2M)}.$$

When x_0 and v_0 are both positive, or both are negative, this value is negative, an unacceptable value.

(b) The equation defining t in part (a) yields only one value; that is, there can be at most one time at which the mass passes through equilibrium. There will be one when the equation yields a positive value for t. This occurs when

$$-\frac{x_0}{v_0 + \beta x_0/(2M)} > 0.$$

When $x_0 > 0$, this requires

$$v_0 + \frac{\beta x_0}{2M} < 0 \quad\Longrightarrow\quad \frac{v_0}{x_0} + \frac{\beta}{2M} < 0.$$

On the other hand when $x_0 < 0$, we must have

$$v_0 + \frac{\beta x_0}{2M} > 0 \quad\Longrightarrow\quad \frac{v_0}{x_0} + \frac{\beta}{2M} < 0.$$

12. (a) The initial-value problem describing the position $x(t)$ of the mass is

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0, \qquad x(0) = x_0, \quad x'(0) = v_0.$$

The auxiliary equation is $Mm^2 + \beta m + k = 0$ with solutions $m = \dfrac{-\beta \pm \sqrt{\beta^2 - 4kM}}{2M}$. Since the motion is overdamped, $\beta^2 - 4kM > 0$, and the auxiliary has real roots. Suppose we denote them by $\omega_1 = \dfrac{-\beta - \sqrt{\beta^2 - 4kM}}{2M}$ and $\omega_2 = \dfrac{-\beta + \sqrt{\beta^2 - 4kM}}{2M}$. A general solution of the differential equation is $x(t) = C_1e^{\omega_1 t} + C_2e^{\omega_2 t}$. To satisfy the initial conditions, we must have

$$x_0 = C_1 + C_2, \quad v_0 = \omega_1 C_1 + \omega_2 C_2 \qquad \Longrightarrow \qquad C_1 = \frac{\omega_2 x_0 - v_0}{\omega_2 - \omega_1}, \quad C_2 = \frac{v_0 - \omega_1 x_0}{\omega_2 - \omega_1}.$$

Thus,

$$x(t) = \left(\frac{\omega_2 x_0 - v_0}{\omega_2 - \omega_1}\right)e^{\omega_1 t} + \left(\frac{v_0 - \omega_1 x_0}{\omega_2 - \omega_1}\right)e^{\omega_2 t} \text{ m}.$$

The mass passes through the equilibrium position when

$$0 = x(t) = \left(\frac{\omega_2 x_0 - v_0}{\omega_2 - \omega_1}\right)e^{\omega_1 t} + \left(\frac{v_0 - \omega_1 x_0}{\omega_2 - \omega_1}\right)e^{\omega_2 t}.$$

This implies that

$$e^{\sqrt{\beta^2 - 4kM}\,t/M} = \frac{1 - \dfrac{\omega_2 x_0}{v_0}}{1 - \dfrac{\omega_1 x_0}{v_0}}.$$

When x_0 and v_0 are both positive, or both are negative, the right side of this equation is between 0 and 1, an unacceptable value.

(b) The equation defining t in part (a) yields only one value; that is, there can be at most one time at which the mass passes through equilibrium. There will be one when the equation yields a positive value for t. This occurs when

$$\frac{1 - \dfrac{\omega_2 x_0}{v_0}}{1 - \dfrac{\omega_1 x_0}{v_0}} > 1.$$

If $1 - \omega_1 x_0/v_0 > 0$, this requires

$$1 - \frac{\omega_2 x_0}{v_0} > 1 - \frac{\omega_1 x_0}{v_0} \qquad \Longrightarrow \qquad \omega_2 < \omega_1,$$

a contradiction. Thus, we must have

$$1 - \frac{\omega_1 x_0}{v_0} < 0 \quad \Longrightarrow \quad \frac{v_0}{x_0} - \omega_1 > 0 \quad \Longrightarrow \quad \frac{\beta + \sqrt{\beta^2 - 4kM}}{2M} + \frac{v_0}{x_0} < 0.$$

13. If x measures displacement of the platform from its equilibrium position, then the differential equation for the combined motion is

$$\left(\frac{W + w}{g}\right)\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0.$$

The auxiliary equation is $\left(\dfrac{W + w}{g}\right)m^2 + \beta m + k = 0$ with solutions

$$m = \frac{-\beta \pm \sqrt{\beta^2 - 4k(W + w)/g}}{2(W + w)/g}.$$

Oscillations occur for large w, and for small values of w no oscillations occur. The largest value of w for no oscillations occurs when

$$\beta^2 - \frac{4k(W+w)}{g} = 0 \quad \Longrightarrow \quad w = \frac{\beta^2 g}{4k} - W.$$

14. The differential equation describing the angle θ of opening of the door as a function of time t is

$$I\frac{d^2\theta}{dt^2} + \beta\frac{d\theta}{dt} + k\theta = 0.$$

The auxiliary equation $Im^2 + \beta m + k = 0$ has roots

$$m = \frac{-\beta \pm \sqrt{\beta^2 - 4kI}}{2I}.$$

Continual motion back and forth is curtailed when motion is critically damped or overdamped, and this occurs when $\beta^2 - 4kI \geq 0$; that is $\beta \geq 2\sqrt{kI}$.

15. Suppose β_u, β_c, and β_o represent damping coefficients for underdamped, critically damped, and overdamped motions. According to equations 5.19, 5.21, and 5.22, displacements for the mass are given by

$$\begin{aligned}
x(t) &= e^{-\beta_u t/(2M)}(C_1 \cos\omega t + C_2 \sin\omega t), \quad \text{(underdamped motion)}\\
x(t) &= (C_1 + C_2 t)e^{-\beta_c t/(2M)}, \quad \text{(critically damped motion)}\\
x(t) &= C_1 e^{(-\beta_o + \sqrt{\beta_o^2 - 4kM})t/(2M)} + C_2 e^{(-\beta_0 - \sqrt{\beta_0^2 - 4kM})t/(2M)} \quad \text{(overdamped motion).}
\end{aligned}$$

The rate at which $x(t)$ goes to zero is determined by the exponential factors

$$e^{-\beta_u t/(2M)}, \qquad e^{-\beta_c t/(2M)}, \qquad \text{and} \qquad e^{(-\beta_o + \sqrt{\beta_o^2 - 4kM})t/(2M)}.$$

Since $\beta_u < \beta_c$, it follows that $e^{-\beta_c t/(2M)} < e^{-\beta_u t/(2M)}$, and therefore the mass returns to its equilibrium position more quickly in critically damped motion than in underdamped motion. We now show that the mass also returns to its equilibrium position more quickly in critically damped motion than in overdamped motion. This is true provided we can show that

$$\beta_o - \sqrt{\beta_o^2 - 4kM} < \beta_c.$$

But we know that $\beta_c = 2\sqrt{kM}$, and therefore we must show that

$$\begin{aligned}
\beta_o - \sqrt{\beta_o^2 - \beta_c^2} &< \beta_c\\
1 - \sqrt{1 - \left(\frac{\beta_c}{\beta_o}\right)^2} &< \frac{\beta_c}{\beta_o}\\
\sqrt{1 - \left(\frac{\beta_c}{\beta_o}\right)^2} &> 1 - \frac{\beta_c}{\beta_o}\\
1 - \left(\frac{\beta_c}{\beta_o}\right)^2 &> 1 - 2\left(\frac{\beta_c}{\beta_o}\right) + \left(\frac{\beta_c}{\beta_o}\right)^2\\
2\left(\frac{\beta_c}{\beta_o}\right)^2 - 2\left(\frac{\beta_c}{\beta_o}\right) &< 0\\
2\left(\frac{\beta_c}{\beta_o}\right)\left(\frac{\beta_c}{\beta_o} - 1\right) &< 0.
\end{aligned}$$

But this is true since $\beta_c < \beta_o$. The rate of return to equilibrium for underdamped motion can be faster or slower than that for overdamped motion. Certainly, we can say that if β_o is very large, then the rate at which the mass returns to its equilibrium position is very slow (since $\beta_o - \sqrt{\beta_o^2 - 4kM}$ is close to zero). As β_o decreases, the rate at which the mass returns to equilibrium increases until it reaches the rate for critical damping. When β_u is very small, the rate at which the mass returns to equilibrium is also small. As β_u increases, the rate increases until it reaches the rate for critical damping. The rates will be the same if

$$\begin{aligned} \beta_u &= \beta_o - \sqrt{\beta_o^2 - 4kM} \\ \beta_o - \beta_u &= \sqrt{\beta_o^2 - \beta_c^2} \\ \beta_o^2 - 2\beta_o\beta_u + \beta_u^2 &= \beta_o^2 - \beta_c^2 \\ \beta_u^2 - 2\beta_o\beta_u + \beta_c^2 &= 0. \end{aligned}$$

If damping coefficients β_u and β_o are such that the last quantity is negative, then the mass will return to equilibrium more quickly for underdamped motion, whereas if this quantity is positive, return is quicker for overdamped motion.

16. (a) The initial-value problem describing the position $x(t)$ of the mass is

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0, \qquad x(0) = x_0, \quad x'(0) = v_0.$$

The auxiliary equation is $Mm^2 + \beta m + k = 0$ with solutions $m = \dfrac{-\beta \pm \sqrt{\beta^2 - 4kM}}{2M}$. Since the motion is underdamped, $\beta^2 - 4kM < 0$, and the auxiliary has equal roots $m = (-\beta \pm \sqrt{4kM - \beta^2}i)/(2M)$. A general solution of the differential equation is

$$x(t) = e^{-\beta t/(2M)}\left[C_1\cos\left(\frac{\sqrt{4kM-\beta^2}}{2M}t\right) + C_2\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t\right)\right].$$

The initial conditions determine values for C_1 and C_2, but we shall not need them. Any function of this form can also be expressed in the form

$$x(t) = Ae^{-\beta t/(2M)}\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t + \phi\right).$$

(b) The times at which the mass passes through equilibrium are defined by the equation

$$0 = x(t) = Ae^{-\beta t/(2M)}\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t + \phi\right) \qquad \Longrightarrow \qquad \sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t + \phi\right) = 0.$$

Hence,

$$\frac{\sqrt{4kM-\beta^2}}{2M}t + \phi = n\pi \qquad \Longrightarrow \qquad t = \frac{2M(n\pi - \phi)}{\sqrt{4kM-\beta^2}},$$

where $n \geq 1$ is an integer. The time interval between successive passes throught the origin is $\dfrac{2M\pi}{\sqrt{4kM-\beta^2}}$.

(c) Times at which the velocity of the mass is equal to zero are given by

$$0 = A\left[-\frac{\beta}{2M}e^{-\beta t/(2M)}\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t + \phi\right) + \frac{\sqrt{4kM-\beta^2}}{2M}e^{-\beta t/(2M)}\cos\left(\frac{\sqrt{4kM-\beta^2}}{2M}t + \phi\right)\right].$$

This simplifies to

$$-\beta\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t + \phi\right) + \sqrt{4kM-\beta^2}\cos\left(\frac{\sqrt{4kM-\beta^2}}{2M}t + \phi\right) = 0,$$

from which

$$\tan\left(\frac{\sqrt{4kM-\beta^2}}{2M}t+\phi\right)=\frac{\sqrt{4kM-\beta^2}}{\beta}.$$

Thus, times at which the velocity is zero are

$$t_n=\frac{2M}{\sqrt{4kM-\beta^2}}\left[\operatorname{Tan}^{-1}\left(\frac{\sqrt{4kM-\beta^2}}{\beta}\right)+n\pi-\phi\right],$$

where $n \geq 1$ is an integer. Depending on values for ϕ and the inverse tangent function, n might start at a value other than 1. It makes no difference to the rest of our discussion. Suppose x_n are the corresponding values for $x(t)$. Consider the ratio

$$\frac{x_n}{x_{n+2}}=\frac{e^{-\beta t_n/(2M)}\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t_n+\phi\right)}{e^{-\beta t_{n+2}/(2M)}\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t_{n+2}+\phi\right)}=e^{\beta(t_{n+2}-t_n)/(2M)}\frac{\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t_n+\phi\right)}{\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t_{n+2}+\phi\right)}.$$

Since $t_{n+2}-t_n=\dfrac{2M}{\sqrt{4kM-\beta^2}}(2\pi)=\dfrac{4M\pi}{\sqrt{4kM-\beta^2}}$, and therefore

$$e^{\beta(t_{n+2}-t_n)/(2M)}=e^{2\beta\pi/\sqrt{4kM-\beta^2}}.$$

Furthermore,

$$\begin{aligned}\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t_{n+2}+\phi\right)&=\sin\left[\operatorname{Tan}^{-1}\left(\frac{\sqrt{4kM-\beta^2}}{\beta}\right)+(n+2)\pi\right]\\&=\sin\left[\operatorname{Tan}^{-1}\left(\frac{\sqrt{4kM-\beta^2}}{\beta}\right)+n\pi\right]\\&=\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t_n+\phi\right).\end{aligned}$$

Thus, $\dfrac{x_n}{x_{n+2}}=e^{2\beta\pi/\sqrt{4kM-\beta^2}}$.

17. Suppose that s is the stretch in the spring when the support is motionless and the mass is at equilibrium on the end of the spring. At this position $ks-Mg=0$, where $g=9.81$. When the support is at position z and the mass has displacement x, the stretch in the spring is $s-x+z$. Consequently, the differential equation for $x(t)$ is

$$M\frac{d^2x}{dt^2}=k(s-x+z)-Mg-\beta\frac{dx}{dt}.$$

This simplifies to

$$M\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=kf(t).$$

18. Suppose that s is the compression in the spring when the support is motionless and the mass is at equilibrium on the end of the spring. At this position $ks-Mg=0$, where $g=9.81$. When the support is at position z and the mass has displacement x, the compression in the spring is $s-x+z$. Consequently, the differential equation for $x(t)$ is

$$M\frac{d^2x}{dt^2}=k(s-x+z)-Mg-\beta\frac{dx}{dt}.$$

This simplifies to

$$M\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=kf(t).$$

19. Suppose that s is the compression in the spring on level road so that the car is at equilibrium on the spring. At this position $ks - Mg = 0$, where $g = 9.81$. When the tire is at position Y and the car has displacement y, the compression in the spring is $s - y + Y$. Consequently, the differential equation for $y(t)$ is

$$M\frac{d^2y}{dt^2} = k(s - y + Y) - Mg - \beta\frac{dy}{dt}.$$

With $ks - Mg = 0$, this simplifies to

$$M\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = kY = kf(x).$$

Since the speed of the car is v, the x-coordinate of the car is $x = vt$ (taking $t = 0$ when the car passes through the origin), and therefore

$$M\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = kf(vt).$$

20. When damping proportional to velocity is taken into account, displacements s must satisfy

$$M\frac{d^2s}{dt^2} = -Mg\sin\theta - \beta\frac{ds}{dt}.$$

When we replace s with $L\theta$, and $\sin\theta$ by θ for small displacements,

$$M\frac{d^2(L\theta)}{dt^2} = -Mg\theta - \beta\frac{d(L\theta)}{dt} \quad\Longrightarrow\quad M\frac{d^2\theta}{dt^2} + \beta\frac{d\theta}{dt} + \frac{Mg}{L}\theta = 0.$$

EXERCISES 5.3

1. The solution is the same to the time and position of the first stop of the mass. During the return trip to the right, the initial-value problem defining the position of the mass is

$$\frac{d^2x}{dt^2} + 16x = -\frac{g}{10}, \qquad x(0.431082) = -0.191663, \quad x'(0.431082) = 0.$$

A general solution of this differential equation is

$$x(t) = C_3 \cos 4t + C_4 \sin 4t - \frac{g}{160}.$$

The initial conditions require

$$\begin{aligned} -0.191663 &= C_3 \cos 4(0.431082) + C_4 \sin 4(0.431082) - \frac{g}{160}, \\ 0 &= -4C_3 \sin 4(0.431082) + 4C_4 \cos 4(0.431082). \end{aligned}$$

The solution is $C_3 = 0.0199344$ and $C_4 = -0.128817$, so that

$$x(t) = 0.0199344 \cos 4t - 0.128817 \sin 4t - \frac{g}{160}.$$

The mass comes to rest for the second time when

$$0 = x'(t) = -4(0.0199344) \sin 4t - 4(0.128817) \cos 4t \quad \Longrightarrow \quad \tan 4t = -\frac{0.128817}{0.0199344}.$$

Thus, $t = -(1/4)\mathrm{Tan}^{-1}(0.128817/0.0199344) + n\pi/4 = -0.354316 + n\pi/4$. Since t must be larger than 0.431082, we choose $n = 2$ in which case $t = 1.216480$. At this time, the position of the mass is

$$x(1.216480) = 0.0199344 \cos 4(1.216480) - 0.128817 \sin 4(1.216480) - \frac{g}{160} = 0.069038 \text{ m}.$$

2. (a) We should first check that the initial stretch in the spring is sufficient to overcome the force of static friction on the mass so that motion does occur. Since the coefficient of static friction is twice that of kinetic friction, it follows that the minimum force that will cause motion is 1 N. At a stretch of 6 cm, the spring force on the mass is $18(6/100) > 1$. Thus, motion will occur. Since the x-component of the force of friction when the mass is moving to the left is 1/2 N, the initial-value problem describing the position $x(t)$ of the mass from the time it starts until it comes to a stop for the first time is

$$\frac{1}{2}\frac{d^2x}{dt^2} + 18x = \frac{1}{2} \quad \Longrightarrow \quad x'' + 36x = 1, \qquad x(0) = 0.06, \quad x'(0) = 0.$$

(b) The auxiliary equation is $m^2 + 36 = 0$ with solutions $m = \pm 6i$, and therefore $x(t) = C_1 \cos 6t + C_2 \sin 6t + 1/36$. To satisfy the initial conditions, we must have $3/50 = C_1 + 1/36$ and $0 = 6C_2$. Thus, $x(t) = (29/900)\cos 6t + 1/36$. Since $v(t) = -(29/150)\sin 6t$, the mass comes to rest for the first time when $6t = \pi$, and at this time, its position is $x = (29/900)\cos\pi + 1/36 = -1/225$. The spring force at this position has magnitude $18(1/225) = 2/25$ N. Since the force of static friction is 1 N, further motion will not occur.

3. (a) We should first check that the initial stretch in the spring is sufficient to overcome the force of static friction on the mass so that motion does occur. Since the coefficient of static friction is twice that of kinetic friction, it follows that the minimum force that will cause motion is 1 N. At a stretch of 25 cm, the spring force on the mass is $18(1/4) > 1$. Thus, motion will occur. Since the x-component of the force of friction when the mass is moving to the left is 1/2 N, the initial-value problem describing the position $x(t)$ of the mass from the time it starts until to a stop for the first time is

$$\frac{1}{2}\frac{d^2x}{dt^2} + 18x = \frac{1}{2} \quad \Longrightarrow \quad x'' + 36x = 1, \qquad x(0) = 0.25 \quad x'(0) = 0.$$

(b) The auxiliary equation is $m^2+36=0$ with solutions $m=\pm 6i$, and therefore $x(t)=C_1\cos 6t+C_2\sin 6t+1/36$. To satisfy the initial conditions, we must have $1/4=C_1+1/36$ and $0=6C_2$. Thus, $x(t)=(2/9)\cos 6t+1/36$. Since $v(t)=(-4/3)\sin 6t$, the mass comes to rest for the first time when $6t=\pi$, and at this time, its position is $x=(2/9)\cos\pi+1/36=-7/36$ m. The spring force at this position has magnitude $18(7/36)=7/2$ N. Since the force of static friction is 1 N, further motion will occur.

4. The initial-value problem describing the position $x(t)$ of the mass from the time it starts until it comes to a stop for the first time is

$$\frac{1}{5}\frac{d^2x}{dt^2}+5x=-\frac{1}{4}\left(\frac{1}{5}\right)g \quad\Longrightarrow\quad x''+25x=-\frac{g}{4}, \qquad x(0)=0, \quad x'(0)=\frac{1}{2}.$$

The auxiliary equation is $m^2+25=0$ with solutions $m=\pm 5i$, and therefore $x(t)=C_1\cos 5t+C_2\sin 5t-g/100$. To satisfy the initial conditions, we must have $0=C_1-g/100$ and $1/2=5C_2$. Thus, $x(t)=(g/100)\cos 5t+(1/10)\sin 5t-g/100$. The mass comes to rest for the first time when

$$0=x'(t)=-\frac{g}{20}\sin 5t+\frac{1}{2}\cos 5t \qquad\Longrightarrow\qquad \tan 5t=\frac{10}{g}.$$

Solutions are $t=(1/5)\mathrm{Tan}^{-1}(10/g)+n\pi/5=0.158998+n\pi/5$, where n is an integer. The first positive solution is $t=0.158998$. The position of the mass at this time is

$$x=\frac{g}{100}\cos 5(0.158998)+\frac{1}{10}\sin 5(0.158998)-\frac{g}{100}=0.0419843 \text{ m}.$$

The spring force at this position has magnitude $5(0.0419843)=0.210$ N. Since the maximum force of static friction is $(1/2)(1/5)g=0.981$, the mass will not move from this position.

5. The initial-value problem describing the position $x(t)$ of the mass from the time it starts until it comes to a stop for the first time is

$$\frac{1}{5}\frac{d^2x}{dt^2}+5x=-\frac{1}{4}\left(\frac{1}{5}\right)g \quad\Longrightarrow\quad x''+25x=-\frac{g}{4}, \qquad x(0)=0, \quad x'(0)=2.$$

The auxiliary equation is $m^2+25=0$ with solutions $m=\pm 5i$, and therefore $x(t)=C_1\cos 5t+C_2\sin 5t-g/100$. To satisfy the initial conditions, we must have $0=C_1-g/100$ and $2=5C_2$. Thus, $x(t)=(g/100)\cos 5t+(2/5)\sin 5t-g/100$. The mass comes to rest for the first time when

$$0=x'(t)=-\frac{g}{20}\sin 5t+2\cos 5t \qquad\Longrightarrow\qquad \tan 5t=\frac{40}{g}.$$

Solutions are $t=(1/5)\mathrm{Tan}^{-1}(40/g)+n\pi/5=0.266059+n\pi/5$, where n is an integer. The first positive solution is $t=0.266059$. The position of the mass at this time is

$$x=\frac{g}{100}\cos 5(0.266059)+\frac{2}{5}\sin 5(0.266059)-\frac{g}{100}=0.313754 \text{ m}.$$

The spring force at this position has magnitude $5(0.313754)=1.57$ N. Since the maximum force of static friction is $(1/2)(1/5)g=0.981$, the mass will move from this position. The initial-value problem describing the position $x(t)$ of the mass until it comes to a stop for the second time is

$$\frac{1}{5}\frac{d^2x}{dt^2}+5x=\frac{9}{20} \quad\Longrightarrow\quad x''+25x=\frac{g}{4}, \qquad x(0)=0.313754, \quad x'(0)=0,$$

where we have re-initiated time as $t=0$ at the start of this motion. A general solution of the differential equation is $x(t)=C_1\cos 5t+C_2\sin 5t+g/100$. To satisfy the initial conditions, we must have $0.313754=C_1+g/100$ and $0=5C_2$. Thus, $x(t)=0.215654\cos 5t+g/100$. The mass comes to rest for the second time when

$$0=x'(t)=-5(0.215654)\sin 5t \qquad\Longrightarrow\qquad t=\frac{n\pi}{5}.$$

The first positive solution is $t = \pi/5$. The position of the mass at this time is

$$x = 0.215654 \cos \pi + \frac{g}{100} = -0.117554 \text{ m}.$$

The spring force at this position has magnitude $5(0.117554) = 0.588$ N. Since this is less than the maximum force of static friction, the mass will not move from this position.

6. The initial-value problem describing the position $x(t)$ of the mass relative to its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 4000x = 3\cos 100t \quad\Longrightarrow\quad x'' + 40\,000x = 30\cos 100t, \qquad x(0) = 0, \quad x'(0) = 10.$$

The auxiliary equation is $m^2 + 40\,000 = 0$ with solutions $m = \pm 200i$. A general solution of the associated homogeneous equation is $x_h(t) = C_1 \cos 200t + C_2 \sin 200t$. Substituting a particular solution of the form $x_p = A\cos 100t + B\sin 100t$ into the differential equation gives

$$(-10\,000A\cos 100t - 10\,000B\sin 100t) + 40\,000(A\cos 100t + 100B\sin 100t) = 30\cos 100t.$$

This implies that $A = 1/1000$ and $B = 0$, so that $x(t) = C_1\cos 200t + C_2 \sin 200t + (1/1000)\cos 100t$. The initial conditions require $0 = C_1 + 1/1000$ and $10 = 200C_2$. Thus, $x(t) = -(1/1000)\cos 200t + (1/20)\sin 200t + (1/1000)\cos 100t$ m. Because displacements are bounded, resonance does not occur.

7. The initial-value problem describing the position $x(t)$ of the mass relative to its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 4000x = 3\cos 200t \quad\Longrightarrow\quad x'' + 40\,000x = 30\cos 200t, \qquad x(0) = 0, \quad x'(0) = 10.$$

The auxiliary equation is $m^2 + 40\,000 = 0$ with solutions $m = \pm 200i$. A general solution of the associated homogeneous equation is $x_h(t) = C_1 \cos 200t + C_2 \sin 200t$. Substituting a particular solution of the form $x_p = At\cos 200t + Bt\sin 200t$ into the differential equation gives

$$\begin{aligned}(-400A\sin 200t - 40\,000At\cos 200t + 400B\cos 200t - 40\,000Bt\sin 200t)&\\ + 40\,000(At\cos 200t + Bt\sin 200t) &= 30\cos 200t.\end{aligned}$$

This implies that $A = 0$ and $B = 3/40$, so that $x(t) = C_1\cos 200t + C_2\sin 200t + (3t/40)\sin 200t$. The initial conditions require $0 = C_1$ and $10 = 200C_2$. Thus, $x(t) = (1/20 + 3t/40)\sin 200t$ m. Because displacements are unbounded, resonance occurs.

8. The initial-value problem describing the position of the mass relative to its equilibrium position is

$$(1)\frac{d^2x}{dt^2} + 64x = 2\sin 4t, \qquad x(0) = 0, \quad x'(0) = 0.$$

The auxiliary equation is $0 = m^2 + 64$ with solutions $m = \pm 8i$. A general solution of the associated homogeneous differential equation is $x_h(t) = C_1\cos 8t + C_2\sin 8t$. A particular solution is of the form $x_p(t) = A\sin 4t + B\cos 4t$. When we substitute this into the differential equation, we obtain

$$(-16A\sin 4t - 16B\cos 4t) + 64(A\sin 4t + B\cos 4t) = 2\sin 4t.$$

This implies that $A = 1/24$ and $B = 0$. A general solution of the differential equation is therefore $x(t) = C_1\cos 8t + C_2\sin 8t + (1/24)\sin 4t$. To satisfy the initial conditions, we must have $0 = C_1$ and $0 = 8C_2 + 1/6$. Thus, $x(t) = -(1/48)\sin 8t + (1/24)\sin 4t$ m. For large t, oscillations are bounded so resonance does not occur.

9. The initial-value problem describing the position of the mass relative to its equilibrium position is

$$(1)\frac{d^2x}{dt^2} + 64x = 2\sin 8t, \qquad x(0) = 0, \quad x'(0) = 0.$$

The auxiliary equation is $0 = m^2 + 64$ with solutions $m = \pm 8i$. A general solution of the associated homogeneous differential equation is $x_h(t) = C_1 \cos 8t + C_2 \sin 8t$. A particular solution is of the form $x_p(t) = At \sin 8t + Bt \cos 8t$. When we substitute this into the differential equation, we obtain

$$= (-64At \sin 8t 16A \cos 8t - 64Bt \cos 8t - 16B \sin 8t) + 64(At \sin 8t + Bt \cos 8t) = 2 \sin 8t.$$

When we equate coefficents of $\sin 8t$ and $\cos 8t$, we get

$$-16B = 2, \qquad 16A = 0.$$

Thus, $x_p(t) = -(t/8) \cos 8t$, and $x(t) = C_1 \cos 8t + C_2 \sin 8t - (t/8) \cos 8t$. To satisfy the initial conditions, we must have $0 = C_1$ and $0 = 8C_2 - 1/8$. Hence, $x(t) = (1/64) \sin 8t - (t/8) \cos 8t$ m. For large t, oscillations are unbounded and resonance occurs.

10. (a) According to equation 5.4, the initial-value problem for motion of the mass is

$$M\frac{d^2x}{dt^2} + kx = kA \sin \omega t, \qquad x(0) = 0, \quad x'(0) = 0.$$

The auxiliary equation is $0 = Mm^2 + k$ with solutions $m = \pm\sqrt{k/M}i$. A general solution of the associated homogeneous differential equation is $x_h(t) = C_1 \cos(\sqrt{k/M}t) + C_2 \sin(\sqrt{k/M}t)$. A particular solution is of the form $x_p(t) = B \sin \omega t + D \cos \omega t$. When we substitute this into the differential equation, we obtain

$$M(-\omega^2 B \sin \omega t - \omega^2 D \cos \omega t) + k(B \sin \omega t + D \cos \omega t) = kA \sin \omega t.$$

When we equate coefficents of $\sin \omega t$ and $\cos \omega t$, we get

$$-\omega^2 MB + kB = kA, \qquad -\omega^2 MD + kD = 0.$$

Thus, $x_p(t) = \left(\dfrac{kA}{k - \omega^2 M}\right) \sin \omega t$, and

$$x(t) = C_1 \cos\sqrt{\frac{k}{M}}t + C_2 \sin\sqrt{\frac{k}{M}}t + \left(\frac{kA}{k - \omega^2 M}\right) \sin \omega t.$$

To satisfy the initial conditions, we must have $0 = C_1$ and $0 = \sqrt{\dfrac{k}{M}}C_2 + \dfrac{k\omega A}{k - \omega^2 M}$. Hence,

$$x(t) = \frac{\omega\sqrt{kM}A}{\omega^2 M - k} \sin\sqrt{\frac{k}{M}}t + \left(\frac{kA}{k - \omega^2 M}\right) \sin \omega t.$$

(b) When $\omega = \sqrt{k/M}$, the particular solution must be taken in the form $x_p(t) = Bt \sin\sqrt{\dfrac{k}{M}}t + Dt \cos\sqrt{\dfrac{k}{M}}t$. Substitution into the differential equation gives

$$kA \sin\sqrt{\frac{k}{M}}t = M\left(2B\sqrt{\frac{k}{M}} \cos\sqrt{\frac{k}{M}}t - \frac{kBt}{M} \sin\sqrt{\frac{k}{M}}t - 2D\sqrt{\frac{k}{M}} \sin\sqrt{\frac{k}{M}}t - \frac{kDt}{M} \cos\sqrt{\frac{k}{M}}t\right)$$
$$+ k\left(Bt \sin\sqrt{\frac{k}{M}}t + Dt \cos\sqrt{\frac{k}{M}}t\right).$$

When we equate coefficents of $\sin \omega t$ and $\cos \omega t$, we get $-2D\sqrt{kM} = kA$ and $2B\sqrt{kM} = 0$. Thus,

$$x(t) = C_1 \cos\sqrt{\frac{k}{M}}t + C_2 \sin\sqrt{\frac{k}{M}}t - \frac{A\sqrt{k/M}t}{2} \cos\sqrt{\frac{k}{M}}t.$$

To satisfy the initial conditions, we must have $0 = C_1$ and $0 = \sqrt{\dfrac{k}{M}}C_2 - \dfrac{A\sqrt{k/M}}{2}$. Hence,

$$x(t) = \frac{A}{2}\sin\sqrt{\frac{k}{M}}t - \frac{A\sqrt{k/M}t}{2}\cos\sqrt{\frac{k}{M}}t.$$

Oscillations are unbounded and resonance occurs.

11. The differential equation describing the position of the mass is $M\frac{d^2x}{dt^2} + kx = A\cos\omega t$. Solutions of the auxiliary equation $Mm^2 + k = 0$ are $m = \pm\sqrt{k/M}i$. Hence, a general solution of the associated homogeneous equation is $x(t) = C_1\cos\sqrt{k/M}t + C_2\sin\sqrt{k/M}t$. Resonance occurs when $\sqrt{k/M} = \omega$.

12. The initial-value problem describing the position $x(t)$ of the mass relative to its equilibrium position is

$$\frac{1}{5}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 10x = 4\sin 10t \quad\Longrightarrow\quad 2x'' + 15x' + 100x = 40\sin 10t, \qquad x(0) = 0, \quad x'(0) = 0.$$

The auxiliary equation is $2m^2 + 15m + 100 = 0$ with solutions $m = (-15 \pm 5\sqrt{23}i)/4$. A general solution of the associated homogeneous equation is

$$x_h(t) = e^{-15t/4}\left(C_1\cos\frac{5\sqrt{23}t}{4} + C_2\sin\frac{5\sqrt{23}t}{4}\right).$$

A particular solution of the differential equation is of the form $x_p(t) = A\sin 10t + B\cos 10t$. When we substitute this into the differential equation, we obtain

$$\begin{aligned} 2(-100A\sin 10t - 100B\cos 10t) + 15(10A\cos 10t - 10B\sin 10t)& \\ + 100(A\sin 10t + B\cos 10t) = 40\sin 10t&. \end{aligned}$$

When we equate coefficients of $\sin 10t$ and $\cos 10t$, we get

$$-200A - 150B + 100A = 40, \qquad -200B + 150A + 100B = 0.$$

The solution is $A = -8/65$ and $B = -12/65$. Hence, a general solution of the differential equation is $x(t) = e^{-15t/4}[C_1\cos(5\sqrt{23}t/4) + C_2\sin(5\sqrt{23}t/4)] - (4/65)(3\cos 10t + 2\sin 10t)$. To satisfy the initial conditions, we must have $0 = C_1 - 12/65$ and $0 = -15C_1/4 + 5\sqrt{23}C_2/4 - 16/13$. These imply that $C_1 = 12/65$ and $C_2 = 20/(13\sqrt{23})$, and therefore

$$x(t) = e^{-15t/4}\left(\frac{12}{65}\cos\frac{5\sqrt{23}t}{4} + \frac{20}{13\sqrt{23}}\sin\frac{5\sqrt{23}t}{4}\right) - \frac{4}{65}(3\cos 10t + 2\sin 10t) \text{ m}.$$

13. (a) The initial-value problem describing the position $x(t)$ of the mass relative to its equilibrium position is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 100x = 2\sin\omega t, \qquad x(0) = 0, \quad x'(0) = 0.$$

The auxiliary equation is $m^2 + 2m + 100 = 0$ with solutions $m = -1 \pm 3\sqrt{11}i$. A general solution of the associated homogeneous equation is $x_h(t) = e^{-t}\left(C_1\cos 3\sqrt{11}t + C_2\sin 3\sqrt{11}t\right)$. A particular solution of the differential equation is of the form $x_p(t) = A\sin\omega t + B\cos\omega t$. When we substitute this into the differential equation, we obtain

$$\begin{aligned} (-\omega^2A\sin\omega t - \omega^2B\cos\omega t) + 2(\omega A\cos\omega t - \omega B\sin\omega t)& \\ + 100(A\sin\omega t + B\cos\omega t) = 2\sin\omega t&. \end{aligned}$$

When we equate coefficients of $\sin\omega t$ and $\cos\omega t$, we get

$$-\omega^2A - 2\omega B + 100A = 2, \qquad -\omega^2B + 2\omega A + 100B = 0.$$

The solution is $A = 2(100-\omega^2)/[(100-\omega^2)^2+4\omega^2]$ and $B = -4\omega/[(100-\omega^2)^2+4\omega^2]$. Hence, a general solution of the differential equation is

$$x(t) = e^{-t}\left(C_1\cos 3\sqrt{11}t + C_2\sin 3\sqrt{11}t\right) + \frac{1}{(100-\omega^2)^2+4\omega^2}[2(100-\omega^2)\sin\omega t - 4\omega\cos\omega t].$$

To satisfy the initial conditions, we must have

$$0 = C_1 - \frac{4\omega}{(100-\omega^2)^2+4\omega^2}, \qquad 0 = -C_1 + 3\sqrt{11}C_2 + \frac{2\omega(100-\omega^2)}{(100-\omega^2)^2+4\omega^2}.$$

These imply that

$$C_1 = \frac{4\omega}{(100-\omega^2)^2+4\omega^2}, \qquad C_2 = \frac{2}{3\sqrt{11}}\left[\frac{\omega(\omega^2-98)}{(100-\omega^2)^2+4\omega^2}\right].$$

The position of the mass is therefore

$$\begin{aligned} x(t) &= e^{-t}\left\{\frac{4\omega}{(100-\omega^2)^2+4\omega^2}\cos 3\sqrt{11}t + \frac{2\sqrt{11}\omega(\omega^2-98)}{33[(100-\omega^2)^2+4\omega^2]}\sin 3\sqrt{11}t\right\} \\ &\quad + \frac{1}{(100-\omega^2)^2+4\omega^2}[2(100-\omega^2)\sin\omega t - 4\omega\cos\omega t] \\ &= \frac{1}{(100-\omega^2)^2+4\omega^2}\left\{e^{-t}\left[4\omega\cos 3\sqrt{11}t + \frac{2\sqrt{11}\omega(\omega^2-98)}{33}\sin 3\sqrt{11}t\right]\right. \\ &\quad \left. + [2(100-\omega^2)\sin\omega t - 4\omega\cos\omega t]\right\} \text{ m.} \end{aligned}$$

(b) Resonance occurs when the amplitude of the steady-state part of the solution, namely,

$$x_p(t) = \frac{1}{(100-\omega^2)^2+4\omega^2}[2(100-\omega^2)\sin\omega t - 4\omega\cos\omega t],$$

is a maximum. The amplitude is

$$A = \frac{1}{(100-\omega^2)^2+4\omega^2}\sqrt{4(100-\omega^2)^2+16\omega^2} = \frac{2}{\sqrt{(100-\omega^2)^2+4\omega^2}}.$$

This is a maximum when the derivative of $(100-\omega^2)^2+4\omega^2$ vanishes,

$$0 = 2(100-\omega^2)(-2\omega)+8\omega \qquad \Longrightarrow \qquad \omega = 7\sqrt{2}.$$

Maximum amplitude is

$$\frac{2}{\sqrt{(100-98)^2+4(98)}} = \frac{\sqrt{11}}{33}\text{ m.}$$

14. (a) Substituting a particular solution of the form $x_p(t) = B\cos\omega t + C\sin\omega t$ into the differential equation

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = A\cos\omega t,$$

gives

$$M(-\omega^2 B\cos\omega t - \omega^2 C\sin\omega t) + \beta(-\omega B\sin\omega t + \omega C\cos\omega t) + k(B\cos\omega t + C\sin\omega t) = A\cos\omega t.$$

When we equate coefficients of $\cos\omega t$ and $\sin\omega t$, we obtain

$$(k - M\omega^2)B + \beta\omega C = A, \qquad -\beta\omega B + (k-M\omega^2)C = 0.$$

The solution of these is $B = \dfrac{A(k - M\omega^2)}{(k - M\omega^2)^2 + \beta^2\omega^2}$, $\quad C = \dfrac{A\beta\omega}{(k - M\omega^2)^2 + \beta^2\omega^2}$. The particular solution is therefore

$$x_p(t) = \frac{A}{(k - M\omega^2)^2 + \beta^2\omega^2}[(k - M\omega^2)\cos\omega t + \beta\omega\sin\omega t].$$

(b) The amplitude of the particular solution is

$$\frac{A}{(k - M\omega^2)^2 + \beta^2\omega^2}\sqrt{(k - M\omega^2)^2 + \beta^2\omega^2} = \frac{A}{\sqrt{(k - M\omega^2)^2 + \beta^2\omega^2}}.$$

It is a maximum when $(k - M\omega^2)^2 + \beta^2\omega^2$ is smallest. To determine the value of ω that yields the minimum, we solve

$$0 = 2(k - M\omega^2)(-2M\omega) + 2\beta^2\omega = 2\omega[-2M(k - M\omega^2) + \beta^2].$$

The nonzero solution is $\omega = \sqrt{k/M - \beta^2/(2M^2)}$. The amplitude at this value of ω is

$$\frac{A}{\sqrt{\left[k - M\left(\dfrac{k}{M} - \dfrac{\beta^2}{2M^2}\right)\right]^2 + \beta^2\left(\dfrac{k}{M} - \dfrac{\beta^2}{2M^2}\right)}} = \frac{2AM}{\beta\sqrt{4kM - \beta^2}}.$$

15. (a) Suppose y measures the distance the mass moves after striking the platform. Then Newton's second law applied to the motion of the mass gives

$$20\frac{d^2y}{dt^2} = -1000y - 10\frac{dy}{dt} + 20g.$$

When we divide by 10 and attach initial displacement and velocity, we obtain the initial-value problem

$$2\frac{d^2y}{dt^2} + \frac{dy}{dt} + 100y = 2g, \qquad y(0) = 0, \quad y'(0) = 2.$$

The auxiliary equation $2m^2 + m + 100 = 0$ has roots $m = (-1 \pm \sqrt{799}i)/4$. Consequently, a general solution of the differential equation is

$$y(t) = e^{-t/4}\left(C_1\cos\frac{\sqrt{799}t}{4} + C_2\sin\frac{\sqrt{799}t}{4}\right) + \frac{g}{50}.$$

The initial conditions require

$$0 = y(0) = C_1 + \frac{g}{50}, \quad 2 = y'(0) = -\frac{C_1}{4} + \frac{\sqrt{799}C_2}{4}.$$

These imply that $C_1 = -g/50$ and $C_2 = (400 - g)/(50\sqrt{799})$, and therefore

$$y(t) = e^{-t/4}\left[-\frac{g}{50}\cos\frac{\sqrt{799}t}{4} + \left(\frac{400 - g}{50\sqrt{799}}\right)\sin\frac{\sqrt{799}t}{4}\right] + \frac{g}{50}.$$

(b) The maximum displacement experienced by the mass occurs when the mass comes to an instantaneous stop for the first time. We therefore set

$$\begin{aligned} 0 = \frac{dy}{dt} &= -\frac{1}{4}e^{-t/4}\left[-\frac{g}{50}\cos\frac{\sqrt{799}t}{4} + \left(\frac{400 - g}{50\sqrt{799}}\right)\sin\frac{\sqrt{799}t}{4}\right] \\ &\quad + e^{-t/4}\left[\frac{\sqrt{799}g}{200}\sin\frac{\sqrt{799}t}{4} + \frac{\sqrt{799}}{4}\left(\frac{400 - g}{50\sqrt{799}}\right)\cos\frac{\sqrt{799}t}{4}\right]. \end{aligned}$$

This equation implies that

$$t = \frac{4}{\sqrt{799}}\mathrm{Tan}^{-1}\left(\frac{2}{\dfrac{400-g}{200\sqrt{799}} - \dfrac{\sqrt{799}g}{200}}\right) = \frac{4}{\sqrt{799}}(-0.9883 + n\pi),$$

where n is an integer. The smallest positive solution occurs for $n = 1$, and for this value of n, $t = 0.3047$ s. The displacement of the mass at this time is $y(0.3047) = 0.51$ m.

16. Suppose the mass of the chain is M so that its mass per unit length is M/a. When the length of chain hanging from the edge of the table is y, gravity acts on this part of the chain, but it accelerates the entire length of chain. Newton's second law gives

$$M\frac{d^2y}{dt^2} = \frac{Mgy}{a}.$$

This differential equation is subject to the initial conditions $y(0) = b$ and $y'(0) = 0$, provided $t = 0$ is taken at the instant motion begins. The differential equation is linear with auxiliary equation $m^2 - g/a = 0 \Longrightarrow m = \pm\sqrt{g/a}$. A general solution is therefore $y(t) = C_1 e^{\sqrt{g/a}t} + C_2 e^{-\sqrt{g/a}t}$. The initial conditions require

$$b = C_1 + C_2, \quad 0 = \sqrt{\frac{g}{a}}C_1 - \sqrt{\frac{g}{a}}C_2 \quad \Longrightarrow \quad C_1 = C_2 = b/2.$$

Thus, $y(t) = \dfrac{b}{2}(e^{\sqrt{g/a}t} + e^{-\sqrt{g/a}t})$. The chain slides off the table when $y = a$ in which case

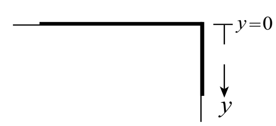
$$a = \frac{b}{2}(e^{\sqrt{g/a}t} + e^{-\sqrt{g/a}t}) \quad \Longrightarrow \quad e^{2\sqrt{g/a}t} - \frac{2a}{b}e^{\sqrt{g/a}t} + 1 = 0.$$

This is a quadratic in $e^{\sqrt{g/a}t}$ with solutions

$$e^{\sqrt{g/a}t} = \frac{2a/b \pm \sqrt{4a^2/b^2 - 4}}{2} = \frac{1}{b}(a \pm \sqrt{a^2 - b^2}) \quad \Longrightarrow t = \sqrt{\frac{a}{g}}\ln\left(\frac{a \pm \sqrt{a^2-b^2}}{b}\right).$$

It is straightforward to verify that $(a - \sqrt{a^2-b^2})/b < 1$ in which case t would be negative, an unacceptable value. Hence, $t = \sqrt{\dfrac{a}{g}}\ln\left(\dfrac{a + \sqrt{a^2-b^2}}{b}\right)$.

17. (a) Suppose the mass of the chain is M so that its mass per unit length is M/a. When the length of chain hanging from the edge of the table is b, the force of gravity on this much chain must be larger than the force of friction on that part of the chain still on the table,



$$\left(\frac{bM}{a}\right)g > \mu_s\left[\frac{(a-b)M}{a}\right]g.$$

Thus, the smallest amount of hanging chain is $b = \mu_s(a - b)$.

(b) When the length of chain hanging from the edge of the table is y, gravity acts on this part of the chain, but it accelerates the entire length of chain. Newton's second law gives

$$M\frac{d^2y}{dt^2} = \frac{Mgy}{a} - \frac{\mu_k Mg}{a}(a-y) \quad \Longrightarrow \quad \frac{d^2y}{dt^2} - \frac{g}{a}(1+\mu_k)y = -\mu_k g.$$

This differential equation is subject to the initial conditions $y(0) = b$ and $y'(0) = 0$, provided $t = 0$ is taken at the instant motion begins. The differential equation is linear with auxiliary equation $m^2 - (g/a)(1+\mu_k) = 0 \implies m = \pm\sqrt{g(1+\mu_k)/a}$. A general solution is therefore $y(t) = C_1 e^{\sqrt{g(1+\mu_k)/a}t} + C_2 e^{-\sqrt{g(1+\mu_k)/a}t} + a\mu_k/(1+\mu_k)$. The initial conditions require

$$b = C_1 + C_2 + \frac{a\mu_k}{1+\mu_k}, \quad 0 = \sqrt{\frac{g(1+\mu_k)}{a}}C_1 - \sqrt{\frac{g(1+\mu_k)}{a}}C_2 \quad \implies \quad C_1 = C_2 = \frac{1}{2}\left(b - \frac{a\mu_k}{1+\mu_k}\right).$$

Thus, $y(t) = \frac{1}{2}\left(b - \frac{a\mu_k}{1+\mu_k}\right)(e^{\sqrt{g(1+\mu_k)/a}t} + e^{-\sqrt{g(1+\mu_k)/a}t}) + \frac{a\mu_k}{1+\mu_k}$. The chain slides off the table when $y = a$,

$$a = \frac{1}{2}\left(b - \frac{a\mu_k}{1+\mu_k}\right)(e^{\sqrt{g(1+\mu_k)/a}t} + e^{-\sqrt{g(1+\mu_k)/a}t}) + \frac{a\mu_k}{1+\mu_k},$$

which can be expressed in the form

$$e^{2\sqrt{g(1+\mu_k)/a}t} - \frac{2a}{b(1+\mu_k) - a\mu_k}e^{\sqrt{g(1+\mu_k)/a}t} + 1 = 0.$$

This is a quadratic in $e^{\sqrt{g(1+\mu_k)/a}t}$ with solutions

$$e^{\sqrt{g(1+\mu_k)/a}t} = \frac{1}{2}\left[\frac{2a}{b(1+\mu_k) - a\mu_k} \pm \sqrt{\frac{4a^2}{[b(1+\mu_k) - a\mu_k]^2} - 4}\right] = \frac{a \pm \sqrt{a^2 - [b(1+\mu_k) - a\mu_k]^2}}{b(1+\mu_k) - a\mu_k},$$

and

$$t = \sqrt{\frac{a}{g(1+\mu_k)}}\ln\left\{\frac{a \pm \sqrt{a^2 - [b(1+\mu_k) - a\mu_k]^2}}{b(1+\mu_k) - a\mu_k}\right\}.$$

It can be shown that the negative root leads to a value $t < 0$. Hence,

$$t = \sqrt{\frac{a}{g(1+\mu_k)}}\ln\left\{\frac{a + \sqrt{a^2 - [b(1+\mu_k) - a\mu_k]^2}}{b(1+\mu_k) - a\mu_k}\right\}.$$

18. Let us use the coordinate system of Figure 5.6 to measure the displacement of the mass. If s is the stretch in the spring at equilibrium, then when the mass is at position x, the stretch is $s - x + f(t)$. Newton's second law for the motion gives

$$\frac{1}{2}\frac{d^2x}{dt^2} = -10\frac{dx}{dt} - \frac{g}{2} + 250[s - x + f(t)].$$

At equilibrium, $-g/2 + 250s = 0$, so that

$$\frac{1}{2}\frac{d^2x}{dt^2} = -10\frac{dx}{dt} + 250[-x + f(t)] \quad \implies \quad \frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 500x = 50\sin 2t,$$

subject to $x(0) = x'(0) = 0$. The auxiliary equation is $m^2 + 20m + 500 = 0$ with solutions $m = -10 \pm 20i$. A general solution of the associated homogeneous equation is therefore $x_h(t) = e^{-10t}(C_1\cos 20t + C_2\sin 20t)$. When we substitute a particular solution of the form $x_p(t) = A\sin 2t + B\cos 2t$ into the differential equation, we obtain

$$(-4A\sin 2t - 4B\cos 2t) + 20(2A\cos 2t - 2B\sin 2t) + 500(A\sin 2t + B\cos 2t) = 50\sin 2t.$$

Equating coeffcients to zero gives

$$496A - 40B = 50, \qquad 40A + 496B = 0,$$

the solution of which is $A = 1550/15,476$ and $B = -125/15\,476$. A general solution of the nonhomogeneous differential equation is

$$x(t) = e^{-10t}(C_1\cos 20t + C_2\sin 20t) + \frac{1550}{15\,476}\sin 2t - \frac{125}{15\,476}\cos 2t.$$

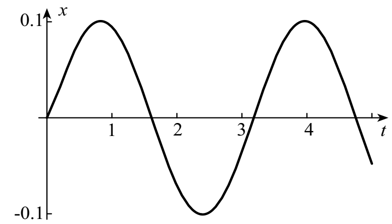
The initial conditions require

$$0 = x(0) = C_1 - \frac{125}{15\,476}, \qquad 0 = x'(0) = -10C_1 + 20C_2 + \frac{1550}{7738}.$$

These give $C_2 = -185/30\,952$. Thus, the position of the mass is given by

$$x(t) = e^{-10t}\left(\frac{125}{15\,476}\cos 20t - \frac{185}{30\,952}\sin 20t\right) + \frac{1550}{15\,476}\sin 2t - \frac{125}{15\,476}\cos 2t.$$

A plot of this function is shown to the right. The damping is so severe that the transient terms disappear almost immediately. The steady-state terms of the particular solution persist forever. The mass oscillates at the same frequency as the motion of the upper support, but with a slightly smaller amplitude, and out of phase with it.



19. (a) Suppose that $s > 0$ represents the compression in the spring when the wheel is at equilibrium. When y is the height of the wheel above its equilibrium position, the stretch (or compression) in the spring is $-s + y - A\sin\pi x$. Newton's second law gives

$$500\frac{d^2y}{dt^2} = -1000(-s + y - A\sin\pi x) - 500g.$$

At equilibrium, $1000s - 500g = 0$, so that the differential equation reduces to

$$500\frac{d^2y}{dt^2} = -1000(y - A\sin\pi x), \qquad \text{or} \qquad \frac{d^2y}{dt^2} + 2y = 2A\sin\pi x.$$

Since the truck is travelling at 18 km/hr or 5m/s, its x-coordinate t seconds after meeting the speed bump is $x = 5t$. Thus,

$$\frac{d^2y}{dt^2} + 2y = 2A\sin 5\pi t, \quad \text{subject to} \quad y(0) = 0, \quad y'(0) = 0.$$

Since it takes the wheel 1/5 of a second to traverse the bump, the equation is in effect only for $0 < t < 1/5$.

(b) The auxiliary equation $m^2 + 2 = 0$ has roots $m = \pm\sqrt{2}i$, and therefore $y_h(t) = C_1\cos\sqrt{2}t + C_2\sin\sqrt{2}t$. When we substitute $y_p(t) = B\sin 5\pi t + D\cos 5\pi t$ into the differential equation,

$$(-25\pi^2 B\sin 5\pi t - 25\pi^2 D\cos 5\pi t) + 2(B\sin 5\pi t + D\cos 5\pi t) = 2A\sin 5\pi t.$$

Equating coefficients of $\sin 5\pi t$ and $\cos 5\pi t$ gives

$$-25\pi^2 B + 2B = 2A, \quad -25\pi^2 D + 2D = 0, \qquad \text{from which} \quad B = \frac{2A}{2 - 25\pi^2}, \quad D = 0.$$

Thus,

$$y(t) = C_1\cos\sqrt{2}t + C_2\sin\sqrt{2}t + \frac{2A}{2 - 25\pi^2}\sin 5\pi t.$$

The initial conditions require

$$0 = y(0) = C_1, \quad 0 = y'(0) = \sqrt{2}C_2 + \frac{10\pi A}{2 - 25\pi^2}.$$

Displacement of the wheel is therefore

$$y(t) = -\frac{5\sqrt{2}\pi A}{2 - 25\pi^2}\sin\sqrt{2}t + \frac{2A}{2 - 25\pi^2}\sin 5\pi t = \frac{A}{2 - 25\pi^2}(2\sin 5\pi t - 5\sqrt{2}\pi\sin\sqrt{2}t)\ \text{m}.$$

20. The initial-value problem describing the position $x(t)$ of the mass from the time it starts until it comes to a stop for the first time is

$$M\frac{d^2x}{dt^2}+kx=-\mu Mg, \qquad x(0)=x_0, \quad x'(0)=v_0.$$

The auxiliary equation is $Mm^2+k=0$ with solutions $m=\pm\sqrt{k/M}i$, and therefore $x(t)=C_1\cos\sqrt{k/M}t+C_2\sin\sqrt{k/M}t-\mu Mg/k$. To satisfy the initial conditions, we must have $x_0=C_1-\mu Mg/k$ and $v_0=\sqrt{k/M}C_2$. Thus,

$$x(t)=\left(x_0+\frac{\mu Mg}{k}\right)\cos\sqrt{\frac{k}{M}}t+\sqrt{\frac{M}{k}}v_0\sin\sqrt{\frac{k}{M}}t-\frac{\mu Mg}{k}.$$

The mass comes to a stop for the first time when

$$0=x'(t)=-\sqrt{\frac{k}{M}}\left(x_0+\frac{\mu Mg}{k}\right)\sin\sqrt{\frac{k}{M}}t+v_0\cos\sqrt{\frac{k}{M}}t.$$

We can rewrite this equation in the form

$$\tan\sqrt{\frac{k}{M}}t=\frac{v_0}{\sqrt{k/M}(x_0+\mu Mg/k)} \qquad \Longrightarrow \qquad t=\sqrt{\frac{M}{k}}\left[\mathrm{Tan}^{-1}\!\left(\frac{v_0\sqrt{M/k}}{x_0+\mu Mg/k}\right)+n\pi\right],$$

where n is an integer. For the smallest positive solution we choose $n=0$.

21. The initial-value problem describing the position $x(t)$ of the mass from the time it starts until it comes to a stop for the first time is

$$M\frac{d^2x}{dt^2}+kx=\mu Mg, \qquad x(0)=x_0, \quad x'(0)=v_0,$$

where $v_0<0$. The auxiliary equation is $Mm^2+k=0$ with solutions $m=\pm\sqrt{k/M}i$, and therefore $x(t)=C_1\cos\sqrt{k/M}t+C_2\sin\sqrt{k/M}t+\mu Mg/k$. To satisfy the initial conditions, we must have $x_0=C_1+\mu Mg/k$ and $v_0=\sqrt{k/M}C_2$. Thus,

$$x(t)=\left(x_0-\frac{\mu Mg}{k}\right)\cos\sqrt{\frac{k}{M}}t+\sqrt{\frac{M}{k}}v_0\sin\sqrt{\frac{k}{M}}t+\frac{\mu Mg}{k}.$$

The mass comes to a stop for the first time when

$$0=x'(t)=-\sqrt{\frac{k}{M}}\left(x_0-\frac{\mu Mg}{k}\right)\sin\sqrt{\frac{k}{M}}t+v_0\cos\sqrt{\frac{k}{M}}t.$$

Except when $x_0=\mu Mg/k$, we can rewrite this equation in the form

$$\tan\sqrt{\frac{k}{M}}t=\frac{v_0}{\sqrt{k/M}(x_0-\mu Mg/k)} \qquad \Longrightarrow \qquad t=\sqrt{\frac{M}{k}}\left[\mathrm{Tan}^{-1}\!\left(\frac{v_0\sqrt{M/k}}{x_0-\mu Mg/k}\right)+n\pi\right],$$

where n is an integer. For the smallest positive solution, we obtain

$$t=\begin{cases}\sqrt{\dfrac{M}{k}}\mathrm{Tan}^{-1}\!\left(\dfrac{v_0\sqrt{M/k}}{x_0-\mu Mg/k}\right), & \text{when } x_0<\mu Mg/k\\[2ex] \sqrt{\dfrac{M}{k}}\dfrac{\pi}{2}, & \text{when } x_0=\mu Mg/k\\[2ex] \sqrt{\dfrac{M}{k}}\left[\mathrm{Tan}^{-1}\!\left(\dfrac{v_0\sqrt{M/k}}{x_0-\mu Mg/k}\right)+\pi\right], & \text{when } x_0>\mu Mg/k.\end{cases}$$

22. According to equation 5.4, the differential equation for displacement $x(t)$ is

$$M\frac{d^2x}{dt^2} + kx = kA\sin\omega t.$$

The auxiliary equation is $Mm^2 + k = 0$ with solutions $m = \pm\sqrt{k/M}i$, so that

$$x_h(t) = C_1\cos\sqrt{\frac{k}{M}}t + C_2\sin\sqrt{\frac{k}{M}}t.$$

Resonance occurs when $\omega = \sqrt{k/M}$. In the nonresonant case, $x_p(t) = D\sin\omega t + E\cos\omega t$. Substitution into the differential equation gives

$$M(-\omega^2 D\sin\omega t - \omega^2 E\cos\omega t) + k(D\sin\omega t + E\cos\omega t) = kA\sin\omega t.$$

When we equate coefficients of $\sin\omega t$ and $\cos\omega t$,

$$-M\omega^2 D + kD = kA, \quad -M\omega^2 E + kE = 0 \qquad \Longrightarrow \qquad D = \frac{kA}{k - M\omega^2}, \quad E = 0.$$

Thus, $x(t) = C_1\cos\sqrt{\dfrac{k}{M}}t + C_2\sin\sqrt{\dfrac{k}{M}}t + \dfrac{kA}{k - M\omega^2}\sin\omega t$. The initial conditions require

$$x_0 = x(0) = C_1, \quad v_0 = x'(0) = C_2\sqrt{\frac{k}{M}} + \frac{kA\omega}{k - M\omega^2}.$$

Displacement of the mass is therefore

$$x(t) = x_0\cos\sqrt{\frac{k}{M}}t + \sqrt{\frac{M}{k}}\left(v_0 - \frac{kA\omega}{k - M\omega^2}\right)\sin\sqrt{\frac{k}{M}}t + \frac{kA}{k - M\omega^2}\sin\omega t.$$

Resonance occurs when $\omega = \sqrt{k/M}$.

23. If $y > 0$ is the depth of the bottom surface of the cube, then Newton's second law from time $t = 0$ when the cube is released until it is completely submerged gives

$$1200\frac{d^2y}{dt^2} = 1200g - 2\frac{dy}{dt} - y(1)^2(1000)g \qquad \Longrightarrow \qquad 600\frac{d^2y}{dt^2} + \frac{dy}{dt} + 500gy = 600g,$$

subject to $y(0) = 0$ and $y'(0) = 0$. The auxiliary equation is

$$600m^2 + m + 500g = 0 \quad \text{with solution} \quad m = \frac{-1 \pm \sqrt{1 - 1\,200\,000g}}{1200}.$$

If we set $\omega = \sqrt{1\,200\,000g - 1}/1200$, then a general solution of the differential equation is

$$y(t) = e^{-t/1200}(C_1\cos\omega t + C_2\sin\omega t) + \frac{6}{5}.$$

The initial conditions require

$$0 = y(0) = C_1 + \frac{6}{5}, \qquad 0 = y'(0) = -\frac{C_1}{1200} + \omega C_2.$$

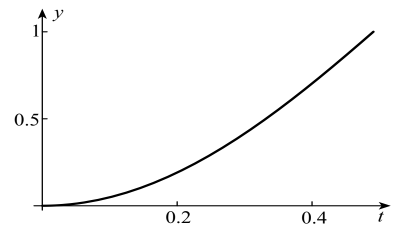
These give $C_1 = -6/5$ and $C_2 = -1/(1000\omega)$, and therefore

$$y(t) = \frac{6}{5} - \frac{e^{-t/1200}}{1000\omega}(1200\omega\cos\omega t + \sin\omega t).$$

This is valid as long as $y \le 1$. When $y = 1$,

$$1 = \frac{6}{5} - \frac{e^{-t/1200}}{1000\omega}(1200\omega\cos\omega t + \sin\omega t),$$

the numerical solution of which is $t = 0.49$ s. A plot of $y(t)$ for $0 \le t \le 0.49$ is shown to the right.



24. If $y > 0$ is the depth of the bottom surface of the cube, then Newton's second law from time $t = 0$ when the cube is released gives

$$500\frac{d^2y}{dt^2} = 500g - 2\frac{dy}{dt} - y(1)^2(1000)g \qquad \Longrightarrow \qquad 250\frac{d^2y}{dt^2} + \frac{dy}{dt} + 500gy = 250g,$$

subject to $y(0) = 0$ and $y'(0) = 0$. The auxiliary equation is

$$250m^2 + m + 500g = 0 \quad \text{with solution} \quad m = \frac{-1 \pm \sqrt{1 - 500\,000g}}{500}.$$

If we set $\omega = \sqrt{500\,000g - 1}/500$, then a general solution of the differential equation is

$$y(t) = e^{-t/500}(C_1 \cos\omega t + C_2 \sin\omega t) + \frac{1}{2}.$$

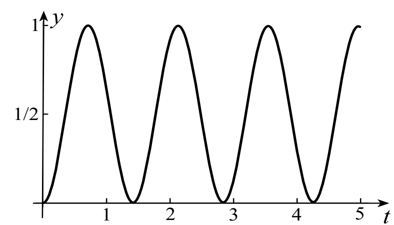
The initial conditions require

$$0 = y(0) = C_1 + \frac{1}{2}, \qquad 0 = y'(0) = -\frac{C_1}{500} + \omega C_2.$$

These give $C_1 = -1/2$ and $C_2 = -1/(1000\omega)$, and therefore

$$y(t) = \frac{1}{2} - \frac{e^{-t/500}}{1000\omega}(500\omega\cos\omega t + \sin\omega t).$$

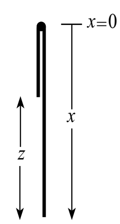
A plot of this function is shown to the right.



25. (a) If x is the length of the longer piece of cable, then Newton's second law for acceleration of the cable is

$$25\rho\frac{d^2x}{dt^2} = 9.81\rho z,$$

where ρ is the mass per unit length of the cable, and z is as shown in the figure to the right. Since $x + (x - z) = 25$, it follows that $z = 2x - 25$ and



$$25\frac{d^2x}{dt^2} = 9.81(2x - 25),$$

or,

$$25\frac{d^2x}{dt^2} - 19.62x = -245.25.$$

The auxiliary equation $25m^2 - 19.62 = 0$ has roots $\pm\sqrt{19.62/25}$. If we denote the positive root by m, then $x(t) = C_1e^{mt} + C_2e^{-mt} + 245.25/19.62$. The initial conditions $x(0) = 15$ and $x'(0) = 0$ require $15 = C_1 + C_2 + 245.25/19.62$ and $0 = mC_1 - mC_2$. These imply that $C_1 = C_2 = 1.25$.

The cable slides off the peg when $25 = 1.25(e^{mt} + e^{-mt}) + 245.25/19.62$ and the solution of this equation is 2.59 s.
(b) In this case Newton's second gives

$$25\rho\frac{d^2x}{dt^2} = 9.81\rho z - 9.81\rho \quad \Longrightarrow \quad 25\frac{d^2x}{dt^2} - 19.62x = -255.06.$$

The solution of this differential equation is $x(t) = C_1e^{mt} + C_2e^{-mt} + 255.06/19.62$, where m is as in part (a). The initial conditions require $15 = C_1 + C_2 + 255.06/19.62$ and $0 = mC_1 - mC_2$, and these gives $C_1 = C_2 = 1$. The cable slides off the peg when $25 = e^{mt} + e^{-mt} + 255.06/19.62$ and the solution of this equation is 2.80 s.

EXERCISES 5.4

1. The initial-value problem describing charge $Q(t)$ on the capacitor is

$$2\frac{d^2Q}{dt^2} + \frac{1}{0.001}Q = 20 \quad \Longrightarrow \quad Q'' + 500Q = 10, \qquad Q(0) = 0, \quad Q'(0) = 0.$$

The auxiliary equation is $0 = m^2 + 500$ with solutions $m = \pm 10\sqrt{5}i$. A general solution of the differential equation is therefore $Q(t) = C_1 \cos 10\sqrt{5}t + C_2 \sin 10\sqrt{5}t + 1/50$. To satisfy the initial conditions, we must have $0 = C_1 + 1/50$ and $0 = 10\sqrt{5}C_2$. Thus, $Q(t) = -(1/50)\cos 10\sqrt{5}t + 1/50$ C, and the current in the circuit is $I(t) = (1/\sqrt{5})\sin 10\sqrt{5}t$ A.

2. The initial-value problem describing charge $Q(t)$ on the capacitor is

$$(1)\frac{d^2Q}{dt^2} + 100\frac{dQ}{dt} + \frac{1}{0.02}Q = 0 \quad \Longrightarrow \quad Q'' + 100Q' + 50Q = 0, \qquad Q(0) = 5 \quad Q'(0) = 0.$$

The auxiliary equation is $m^2 + 100m + 50 = 0$ with solutions $m = -50 \pm 35\sqrt{2}$. A general solution of the differential equation is therefore $Q(t) = C_1 e^{(-50+35\sqrt{2})t} + C_2 e^{-(50+35\sqrt{2})t}$. To satisfy the initial conditions, we must have $5 = C_1 + C_2$ and $0 = (-50 + 35\sqrt{2})C_1 - (50 + 35\sqrt{2})C_2$. These imply that $C_1 = 5(5\sqrt{2} + 7)/14$ and $C_2 = 5(7 - 5\sqrt{2})/14$, and therefore

$$Q(t) = \frac{5}{14}(5\sqrt{2} + 7)e^{(-50+35\sqrt{2})t} + \frac{5}{14}(7 - 5\sqrt{2})e^{-(50+35\sqrt{2})t} \text{ C}.$$

3. The initial-value problem describing the current $I(t)$ in the circuit is

$$5\frac{d^2I}{dt^2} + 20\frac{dI}{dt} = 20\cos 2t, \qquad I(0) = 0, \quad I'(0) = 0.$$

The auxiliary equation is $5m^2 + 20m = 0$ with solutions $m = 0, -4$. A general solution of the associated homogeneous differential equation is therefore $I(t) = C_1 + C_2 e^{-4t}$. Substituting a particular solution of the form $I_p(t) = A\cos 2t + B\sin 2t$ into the differential equation gives

$$5(-4A\cos 2t - 4B\sin 2t) + 20(-2A\sin 2t + 2B\cos 2t) = 20\cos 2t.$$

This implies that $-20A + 40B = 20$ and $-20B - 40A = 0$, from which $A = -1/5$ and $B = 2/5$. The current is therefore $I(t) = C_1 + C_2 e^{-4t} + (2\sin 2t - \cos 2t)/5$. The initial conditions require $0 = C_1 + C_2 - 1/5$ and $0 = -4C_2 + 4/5$, from which $C_1 = 0$ and $C_2 = 1/5$. The transient part of the current is $(1/5)e^{-4t}$ A, and the steady-state part is $(2\sin 2t - \cos 2t)/5$ A.

4. The initial-value problem describing charge $Q(t)$ on the capacitor is

$$\frac{1}{2}\frac{d^2Q}{dt^2} + 3\frac{dQ}{dt} + \frac{1}{0.1}Q = 0 \quad \Longrightarrow \quad Q'' + 6Q + 20Q = 0, \qquad Q(0) = 0, \quad Q'(0) = 1.$$

The auxiliary equation $m^2 + 6m + 20 = 0$ has solutions $m = -3 \pm \sqrt{11}i$. A general solution of the differential equation is therefore $Q(t) = e^{-3t}(C_1 \cos\sqrt{11}t + C_2 \sin\sqrt{11}t)$. To satisfy the initial conditions, we must have $0 = C_1$ and $1 = -3C_1 + \sqrt{11}C_2$. Thus, $Q(t) = (1/\sqrt{11})e^{-3t}\sin\sqrt{11}t$. To find the maximum charge on the capacitor, we find critical points for $Q(t)$,

$$0 = Q'(t) = \frac{1}{\sqrt{11}}(-3e^{-3t}\sin\sqrt{11}t + \sqrt{11}e^{-3t}\cos\sqrt{11}t).$$

The smallest positive solution of this equation is $t = (1/\sqrt{11})\mathrm{Tan}^{-1}(\sqrt{11}/3)$, and the charge on the capacitor at this time is 0.105 C.

5. The initial-value problem describing the current in the circuit is

$$\frac{25}{9}\frac{d^2I}{dt^2} + \frac{1}{0.04}I = -45\sin 3t \quad \Longrightarrow \quad 5I'' + 45I = -81\sin 3t, \qquad I(0) = I'(0) = 0.$$

The auxiliary equation $5m^2 + 45 = 0$ has solutions $m = \pm 3i$, and therefore $I_h(t) = C_1 \cos 3t + C_2 \sin 3t$. A particular solution is of the form $x_p(t) = At \sin 3t + Bt \cos 3t$. When we substitute this into the differential equation, we get

$$5(6A \cos 3t - 9At \sin 3t - 6B \sin 3t - 9Bt \cos 3t) + 45(At \sin 3t + Bt \cos 3t) = -81 \sin 3t.$$

This implies that $A = 0$ and $B = 27/10$. Thus, $I(t) = C_1 \cos 3t + C_2 \sin 3t + (27t/10) \cos 3t$. To satisfy the initial conditions, we must have $0 = C_1$ and $0 = 3C_2 + 27/10$, and the solution becomes $I(t) = -(9/10) \sin 3t + (27/10)t \cos 3t$ A. Since the current becomes unbounded, resonance does occur.

EXERCISES 5.5

1. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\left(\frac{-9.81m}{L}\right), \qquad y(0) = y''(0) = 0, \quad y(L) = y''(L) = 0.$$

Four antidifferentiations of this equation give

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 - \frac{9.81mx^4}{24EIL}.$$

The boundary conditions require

$$0 = C_1, \quad 0 = 2C_3, \quad 0 = C_1 + C_2L + C_3L^2 + C_4L^3 - \frac{9.81mL^3}{24EI}, \quad 0 = 2C_3 + 6C_4L - \frac{9.81mL}{2EI}.$$

These imply that $C_2 = -\dfrac{9.81mL^2}{24EI}$ and $C_4 = \dfrac{9.81m}{12EI}$, and deflections of the beam are

$$y(x) = -\frac{9.81mL^2x}{24EI} + \frac{9.81mx^3}{12EI} - \frac{9.81mx^4}{24EIL} = -\frac{9.81m}{24EIL}(x^4 - 2Lx^3 + L^3x).$$

2. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\left(\frac{-9.81m}{L}\right), \qquad y(0) = y'(0) = 0, \quad y(L) = y'(L) = 0.$$

Four antidifferentiations of this equation give

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 - \frac{9.81mx^4}{24EIL}.$$

The boundary conditions require

$$0 = C_1, \quad 0 = C_2, \quad 0 = C_1 + C_2L + C_3L^2 + C_4L^3 - \frac{9.81mL^3}{24EI}, \quad 0 = C_2 + 2C_3L + 3C_4L^2 - \frac{9.81mL^2}{6EI}.$$

These imply that $C_3 = -\dfrac{9.81mL}{24EI}$ and $C_4 = \dfrac{9.81m}{12EI}$, and deflections of the beam are

$$y(x) = -\frac{9.81mLx^2}{24EI} + \frac{9.81mx^3}{12EI} - \frac{9.81mx^4}{24EIL} = -\frac{9.81m}{24EIL}(x^4 - 2Lx^3 + L^2x^2).$$

3. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\left(\frac{-9.81m}{L}\right), \qquad y(0) = y'(0) = 0, \quad y(L) = y''(L) = 0.$$

Four antidifferentiations of this equation give

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 - \frac{9.81mx^4}{24EIL}.$$

The boundary conditions require

$$0 = C_1, \quad 0 = C_2, \quad 0 = C_1 + C_2L + C_3L^2 + C_4L^3 - \frac{9.81mL^3}{24EI}, \quad 0 = 2C_3 + 6C_4L - \frac{9.81mL}{2EI}.$$

These imply that $C_3 = -\dfrac{9.81mL}{16EI}$ and $C_4 = \dfrac{5(9.81)m}{48EI}$, and deflections of the beam are

$$y(x) = -\frac{9.81mLx^2}{16EI} + \frac{5(9.81)mx^3}{48EI} - \frac{9.81mx^4}{24EIL} = -\frac{9.81m}{48EIL}(3L^2x^2 - 5Lx^3 + 2x^4).$$

4. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\left\{\frac{-9.81m}{L} - \frac{19.62M}{L}[h(x) - h(x - L/2)]\right\}, \qquad y(0) = y'(0) = 0, \quad y(L) = y''(L) = 0.$$

Since $h(x) = 1$ for $0 < x < L$, four integrations of this equation give

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 - \frac{9.81mx^4}{24EIL} - \frac{9.81M}{12EIL}[x^4 - (x - L/2)^4h(x - L/2)].$$

The boundary conditions require

$$0 = C_1, \quad 0 = C_2, \quad 0 = C_1 + C_2L + C_3L^2 + C_4L^3 - \frac{9.81mL^3}{24EI} - \frac{9.81M}{12EIL}[L^4 - (L/2)^4],$$
$$0 = 2C_3 + 6C_4L - \frac{9.81mL}{2EI} - \frac{9.81M}{EIL}[L^2 - (L/2)^2].$$

These imply that

$$C_3 = -\frac{9.81mL}{16EI} - \frac{9(9.81)ML}{128EI}, \qquad C_4 = \frac{5(9.81)m}{48EI} + \frac{19(9.81)M}{128EI}.$$

Deflections of the beam are

$$\begin{aligned} y(x) &= \left[-\frac{9.81mL}{16EI} - \frac{9(9.81)ML}{128EI}\right]x^2 + \left[\frac{5(9.81)m}{48EI} + \frac{19(9.81)M}{128EI}\right]x^3 \\ &\qquad - \frac{9.81mx^4}{24EIL} - \frac{9.81M}{12EIL}\left[x^4 - (x - L/2)^4h(x - L/2)\right] \\ &= \frac{9.81}{384EIL}\big\{-(24m + 27M)L^2x^2 + (40m + 57M)Lx^3 \\ &\qquad - 16mx^4 - 32M\big[x^4 - (x - L/2)^4h(x - L/2)\big]\big\}. \bullet \end{aligned}$$

5. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\left\{\frac{-9.81m}{L} - \frac{19.62M}{L}[h(x - L/4) - h(x - 3L/4)]\right\}, \qquad y(0) = y'(0) = 0, \quad y(L) = y''(L) = 0.$$

Four integrations of this equation give

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 - \frac{9.81mx^4}{24EIL} - \frac{9.81M}{12EIL}[(x - L/4)^4h(x - L/4) - (x - 3L/4)^4h(x - 3L/4)].$$

The boundary conditions require

$$0 = C_1, \quad 0 = C_2, \quad 0 = C_1 + C_2L + C_3L^2 + C_4L^3 - \frac{9.81mL^3}{24EI} - \frac{9.81M}{12EIL}[(3L/4)^4 - (L/4)^4],$$
$$0 = 2C_3 + 6C_4L - \frac{9.81mL}{2EI} - \frac{9.81M}{EIL}[(3L/4)^2 - (L/4)^2].$$

These imply that

$$C_3 = -\frac{9.81mL}{16EI} - \frac{11(9.81)ML}{128EI}, \qquad C_4 = \frac{5(9.81)m}{48EI} + \frac{43(9.81)M}{384EI}.$$

Deflections of the beam are

$$\begin{aligned} y(x) &= \left[-\frac{9.81mL}{16EI} - \frac{11(9.81)ML}{128EI}\right]x^2 + \left[\frac{5(9.81)m}{48EI} + \frac{43(9.81)M}{384EI}\right]x^3 \\ &\qquad - \frac{9.81mx^4}{24EIL} - \frac{9.81M}{12EIL}[(x - L/4)^4h(x - L/4) - (x - 3L/4)^4h(x - 3L/4)] \\ &= \frac{9.81}{384EIL}\big\{-(24m + 33M)L^2x^2 + (40m + 43M)Lx^3 \\ &\qquad - 16mx^4 - 32M[(x - L/4)^4h(x - L/4) - (x - 3L/4)^4h(x - 3L/4)]\big\}. \end{aligned}$$

6. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\left[\frac{-9.81m}{L} - \frac{19.62M}{L}h(x - L/2)\right], \qquad y(0) = y'(0) = 0, \quad y''(L) = y'''(L) = 0.$$

Four integrations of this equation give

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 - \frac{9.81mx^4}{24EIL} - \frac{9.81M}{12EIL}(x - L/2)^4h(x - L/2).$$

The boundary conditions require

$$0 = C_1, \quad 0 = C_2, \quad 0 = 2C_3 + 6C_4L - \frac{9.81mL}{2EI} - \frac{9.81M}{EIL}(L/2)^2,$$
$$0 = 6C_4 - \frac{9.81m}{EI} - \frac{19.62M}{EIL}(L/2).$$

These imply that

$$C_3 = -\frac{9.81mL}{4EI} - \frac{3(9.81)ML}{8EI}, \qquad C_4 = \frac{9.81m}{6EI} + \frac{9.81M}{6EI}.$$

Deflections of the beam are

$$\begin{aligned} y(x) &= \left[-\frac{9.81mL}{4EI} - \frac{3(9.81)ML}{8EI}\right]x^2 + \left[\frac{9.81m}{6EI} + \frac{9.81M}{6EI}\right]x^3 \\ &\quad - \frac{9.81mx^4}{24EIL} - \frac{9.81M}{12EIL}(x - L/2)^4h(x - L/2) \\ &= \frac{9.81}{24EIL}\left[-3(2m + 3M)L^2x^2 + 4(m + M)Lx^3 - mx^4 - 2M(x - L/2)^4h(x - L/2)\right]. \end{aligned}$$

The deflection of the right end of the board is

$$\begin{aligned} y(L) &= \frac{9.81}{24EIL}\left[-3(2m + 3M)L^4 + 4(m + M)L^4 - mL^4 - 2M(L - L/2)^4\right] \\ &= -\frac{9.81L^3(24m + 41M)}{192EI}. \end{aligned}$$

7. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\left\{\frac{-9.81m}{L} - \frac{19.62M}{L}[h(x) - h(x - L/2)]\right\}, \qquad y(0) = y'(0) = 0, \quad y''(L) = y'''(L) = 0.$$

Since $h(x) = 1$ for $0 < x < L$, four integrations of this equation give

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 - \frac{9.81mx^4}{24EIL} - \frac{9.81M}{12EIL}[x^4 - (x - L/2)^4h(x - L/2)].$$

The boundary conditions require

$$0 = C_1, \quad 0 = C_2, \quad 0 = 2C_3 + 6C_4L - \frac{9.81mL}{2EI} - \frac{9.81M}{EIL}[L^2 - (L/2)^2],$$
$$0 = 6C_4 - \frac{9.81m}{EI} - \frac{19.62M}{EIL}[L - (L/2)].$$

These imply that

$$C_3 = -\frac{9.81mL}{4EI} - \frac{9.81ML}{8EI}, \qquad C_4 = \frac{9.81m}{6EI} + \frac{9.81M}{6EI}.$$

Deflections of the beam are

$$\begin{aligned}
y(x) &= -\left(\frac{9.81mL}{4EI} + \frac{9.81ML}{8EI}\right)x^2 + \left(\frac{9.81m}{6EI} + \frac{9.81M}{6EI}\right)x^3 \\
&\quad - \frac{9.81mx^4}{24EIL} - \frac{9.81M}{12EIL}[x^4 - (x - L/2)^4 h(x - L/2)] \\
&= \frac{9.81}{24EIL}\{-3(2m + M)L^2x^2 + 4(m + M)Lx^3 - mx^4 - 2M[x^4 - (x - L/2)^4 h(x - L/2)]\}.
\end{aligned}$$

The deflection of the right end of the board is

$$\begin{aligned}
y(L) &= \frac{9.81}{24EIL}\{-3(2m + M)L^4 + 4(m + M)L^4 - mL^4 - 2M[L^4 - (L/2)^4]\} \\
&= -\frac{9.81L^3(24m + 7M)}{192EI}.
\end{aligned}$$

This is less than the deflection in Exercise 6.

8. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\left\{\frac{-9.81m}{L} - \frac{19.62M}{L}[h(x - L/4) - h(x - 3L/4)]\right\}, \qquad y(0) = y'(0) = 0, \quad y''(L) = y'''(L) = 0.$$

Four integrations of this equation give

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 - \frac{9.81mx^4}{24EIL} - \frac{9.81M}{12EIL}[(x - L/4)^4 h(x - L/4) - (x - 3L/4)^4 h(x - 3L/4)].$$

The boundary conditions require

$$\begin{aligned}
&0 = C_1, \quad 0 = C_2, \quad 0 = 2C_3 + 6C_4L - \frac{9.81mL}{2EI} - \frac{9.81M}{EIL}[(3L/4)^2 - (L/4)^2], \\
&0 = 6C_4 - \frac{9.81m}{EI} - \frac{19.62M}{EIL}[(3L/4) - (L/4)].
\end{aligned}$$

These imply that

$$C_3 = -\frac{9.81mL}{4EI} - \frac{9.81ML}{4EI}, \qquad C_4 = \frac{9.81m}{6EI} + \frac{9.81M}{6EI}.$$

Deflections of the beam are

$$\begin{aligned}
y(x) &= \left(-\frac{9.81mL}{4EI} - \frac{9.81ML}{4EI}\right)x^2 + \left(\frac{9.81m}{6EI} + \frac{9.81M}{6EI}\right)x^3 \\
&\quad - \frac{9.81mx^4}{24EIL} - \frac{9.81M}{12EIL}[(x - L/4)^4 h(x - L/4) - (x - 3L/4)^4 h(x - 3L/4)] \\
&= \frac{9.81}{24EIL}\{-6(m + M)L^2x^2 + 4(m + M)Lx^3 - mx^4 \\
&\qquad - 2M[(x - L/4)^4 h(x - L/4) - (x - 3L/4)^4 h(x - 3L/4)]\}.
\end{aligned}$$

The deflection of the right end of the board is

$$\begin{aligned}
y(L) &= \frac{9.81}{24EIL}\{-6(m + M)L^4 + 4(m + M)L^4 - mL^4 - 2M[(L - L/4)^4 - (L - 3L/4)^4]\} \\
&= -\frac{9.81L^3(24m + 21M)}{192EI}.
\end{aligned}$$

This is less than the deflection in Exercise 6, but more than that in Exercise 7.