LINEAR SPACES

FOR PHYSICISTS

Donald W. Trim

The University of Manitoba

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CHAPTER 1 VECTOR SPACES

§1.1 Introduction

A first course in linear algebra identifies various types of matrices and how they are algebraically combined, (added, subtracted, multiplied, and multiplied by scalars), uses Gaussian and Gauss-Jordan elimination to solve systems of linear equations, evaluates determinants, and finds inverse matrices. We assume that the reader is familiar with each of these topics. Toward the end of the first course, there may be a chapter on eigenvalues and eigenvectors of linear transformations and/or matrices, and perhaps even an introduction to *vector spaces.* We do not assume that the reader has studied these topics. We begin our studies of linear algebra with vector spaces, followed by general discussions on eigenvalues and eigenvectors.

Many structures with which the reader is already familiar are examples of vector spaces. For instance, geometric vectors in xyz-space are directed line segments. They are often written in the form $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, where v_x, v_y, and v_z are their x-, y-, and z-components, or, alternatively, in the form $\mathbf{v} = \langle v_x, v_y, v_z \rangle$. We shall see that the set of all such vectors constitutes a vector space. Likewise, the set of all directed line segments in the xy-plane constitutes a vector space.

Many other sets of entities are vector spaces, but they bear no resemblance to geometric vectors. For instance, the set of all m by n matrices, the set of all solutions to a linear, homogeneous differential equation of order n, the set of all polynomials of degree less than or equal some integer n, the set of all convergent sequences, and the set of functions that are continuously differentiable on the interval $a < x < b$ are all vector spaces.

We assume that the reader is familiar with geometry in the xy-plane and xyz-space. For instance, the reader is expected to be able to find vector and scalar equations of planes in space; vector, parametric, and symmetric equations of lines in space; and distances between points, lines and planes. Many of these topics depend on dot and cross products. Not all vector spaces are equipped with these operations, in particular a cross product. A vector space equipped with an *inner product*, the generalization of the dot product, is called an *inner product space*. The length of a geometric vector is the square root of the dot product of the vector with itself $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. A vector space equipped with a norm is called a *normed space*. It may seem strange that some of the vector spaces mentioned in the previous paragraph turn out to be inner product and normed spaces.

Perhaps the biggest difficulty for writer and reader of linear algebra is notation. The writer must avoid notation that is so arcane that subject matter becomes obscure; we hope that our notation is sufficiently suggestive that fundamental ideas of linear algebra are transparent to the reader. Should the reader be sufficiently venturesome to seek out other texts on linear algebra in order to compare presentations, be prepared to see different notations and nomenclature. The same entity may given a different notation by other authors, and even a different name.

§1.2 Definition of a Vector Space

Suppose V is a set of elements, called **vectors**, and S is a set of elements, called **scalars**. In order for V to be a vector space, it must be possible to add vectors, and multiply them by scalars. In other words, for any two vectors $\mathbf{u}$ and $\mathbf{v}$ in V, and any scalar a in S, $\mathbf{u}+\mathbf{v}$ and $a\mathbf{u}$ must be defined and be vectors in V. When a set of vectors satisfies this property, it is said to be **closed under vector addition and scalar multiplication**.

In order to appreciate the requirements of a vector space, convince yourself that geometric vectors (directed line segments) in the xy-plane and xyz-space satisfy the the properties in the following definition.

Definition 1.1 A set V of vectors is said to be a **vector space over a set of scalars** S if it is closed under vector addition and scalar multiplication, and if for any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and any scalars a and b in S, the following properties are valid:

(1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ (vector addition is commutative)

(2) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ (vector addition is associative)

(3) There exists a vector, denoted by $\mathbf{0}$, called the zero vector, such that $\mathbf{0}+\mathbf{v}=\mathbf{v}$.

(4) For every vector $\mathbf{v}$, there exists a vector, denoted by $-\mathbf{v}$, called the additive inverse, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.

(5) $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$ (distributivity of scalar over vector addition)

(6) $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$ (distributivity of vector over scalar addition)

(7) $(ab)\mathbf{v}=a(b\mathbf{v})$

(8) There exists a scalar, denoted by 1, called the multiplicative identity, such that $1\mathbf{v}=\mathbf{v}$.

When the set of scalars S is the set $\mathcal{R}$ of real numbers, V is said to be a **real vector space**; when S is the set of complex number $\mathcal{C}$, V is said to be a **complex vector space**. Other possibilities for S might be the set of integers or the set of rational numbers.

The set of all geometric vectors in the xy-plane with the usual definitions of vector addition and scalar multiplication satisfies the conditions of Definition 1.1, and therefore constitutes a vector space. We denote it by $\mathcal{G}^2$. Likewise, the set of geometric vectors in xyz-space is a vector space denoted by $\mathcal{G}^3$. In early studies of geometric vectors, it is emphasized that the position of the tail of a vector is optional; what is important is the length and direction of the vector, not its placement. In the context of vector spaces, it is best to draw vectors with their tails at the origin, and henceforth, we make this agreement.

Closely related to $\mathcal{G}^2$ is a vector space denoted by $\mathcal{R}^2$. Vectors in $\mathcal{R}^2$ are ordered pairs (x_1,x_2) of real numbers x_1 and x_2, and they are added and multiplied by scalars as follows:

$$(x_1,x_2)+(y_1,y_2)=(x_1+y_1,x_2+y_2), \quad a(x_1,x_2)=(ax_1,ax_2). \tag{1.1}$$

This is exactly the way components of vectors in $\mathcal{G}^2$ are added and multiplied by scalars. In other words, vectors in $\mathcal{G}^2$ and vectors in $\mathcal{R}^2$ are operationally equivalent; it is just the way in which we regard vectors that differs. Vectors in $\mathcal{G}^2$ are directed line segments; vectors in $\mathcal{R}^2$ are ordered pairs. We distinguish between them with angle-brackets surrounding components of vectors in $\mathcal{G}^2$ (such as $\langle 1,-2\rangle$), and parentheses surrounding vectors in $\mathcal{R}^2$ (such as $(1,-2)$).

In similar fashion, we denote by $\mathcal{R}^3$, the vector space of ordered triples (x_1,x_2,x_3) of real numbers, with vector addition and scalr multiplication defined as in equation 1.1, but with a third entry,

$$(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3), \quad a(x_1,x_2,x_3)=(ax_1,ax_2,ax_3). \tag{1.2}$$

Operationally, it is equivalent to $\mathcal{G}^3$. The space of ordered n-tuples $(x_1,x_2,\ldots,x_n)$ is denoted by $\mathcal{R}^n$; it has its geometric counterpart in $\mathcal{G}^n$, but we do have difficulty visualizing directed line segments in this space.

Vectors in $\mathcal{R}^2$ and $\mathcal{R}^3$ are pairs and triples of real numbers, nothing more. But because they are operationally equivalent to vectors in $\mathcal{G}^2$ and $\mathcal{G}^3$, geometric properties of vectors in $\mathcal{G}^2$ and $\mathcal{G}^3$ have corresponding algebraic properties in $\mathcal{R}^2$ and $\mathcal{R}^3$. Conversely, algebraic properties in $\mathcal{R}^2$ and $\mathcal{R}^3$ are reflected by geometric properties in $\mathcal{G}^2$ and $\mathcal{G}^3$. We make use of this in the following way. When we want to discuss a new concept in an arbitrary vector space, we often do so first in $\mathcal{G}^2$ and/or $\mathcal{G}^3$. With a geometric grasp of the idea, it is then straightforward to transfer calculations to tuples in $\mathcal{R}^2$ and $\mathcal{R}^3$, and on to $\mathcal{R}^n$, spaces without geometry. The final transition is to completely arbitrary vector spaces. For instance, the numbers v_x, v_y, and v_z for the geometric vector $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ in $\mathcal{G}^3$ are called the x-, y-, and z-components of the vector. The numbers x_1, x_2, and x_3 for the vector (x_1, x_2, x_3) in $\mathcal{R}^3$ are called the first, second, and third components of the vector. We will eventually learn how to assign components to vectors in every vector space.

Vector space $\mathcal{R}^n$ can be generalized to a complex vector space by using complex numbers as scalars and complex n-tuples. It is denoted by $\mathcal{C}^n$. Other possibilities for vector spaces are illustrated below. The reader is asked to verify the assertions in the exercises.

(1) The set of all m by n real matrices, denoted by $M_{m,n}(\mathcal{R})$, with the usual definition of vector addition and scalar multiplication by reals is a real vector space.

(2) The set of all m by n complex matrices, denoted by $M_{m,n}(\mathcal{C})$, with the usual definition of vector addition and scalar multiplication by complex numbers is a complex vector space.

(3) The set of all real n-tuples with the usual scalar multiplication by complex numbers is not a vector space.

(4) It might seem unusual to consider the set of all complex n-tuples with scalar multiplication by real numbers, but it would then be a real vector space, not the vector space $\mathcal{C}^n$. The adjective "real" identifies the set of scalars, not the set of vectors.

(5) With the usual definitions of addition and scalar multiplication of functions,

$$(f+g)(x) = f(x) + g(x), \qquad (kf)(x) = kf(x),$$

the set of real-valued functions that have continuous n^{th} derivatives on the interval $a \le x \le b$ is a real vector space. It is denoted by $C^n[a,b]$. In the event that the interval is open, $a < x < b$, the space is denoted by $C^n(a,b)$. The vector space of functions with derivatives of all orders on $a \le x \le b$ is denoted by $C^\infty[a,b]$. The space of continuous functions on $a \le x \le b$ is denoted by $C^0[a,b]$.

(6) With the usual definitions of polynomial addition and multiplication by a scalar, the set of real polynomials of degree less than or equal to a fixed integer $n \ge 0$ is a real vector space. The space of real polynomials of only one variable x is denoted by $P_n(x)$. The space of real polynomials of degree less than or equal to n in x and y is denoted by $P_n(x,y)$. The set of all real polynomials of all degrees in x is a vector space denoted by $P(x)$.

(7) The set of complex polynomials of degree less than or equal to a fixed integer $n \ge 0$ is a complex vector space. It is denoted by $P_n(z)$.

Example 1.1 Let $y_1(x), \ldots, y_m(x)$ be m real solutions of the linear, n^{th}-order, homogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

on some interval I where $a_n(x) \ne 0$. A linear combination of these solutions is a function of the form

$$y(x) = a_1y_1(x) + a_2y_2(x) + \cdots + a_my_m(x),$$

where a_1, a_2, …, a_m are real constants. Is the set of all linear combinations a vector space?

Solution Because the differential equation is linear and homogeneous, the superposition principle states that any linear combination of solutions is also a solution. In other words, the set of solutions is closed under vector addition and scalar multiplication. It is straightforward to check that the conditions of Definition 1.1 are satisfied. Consequently, the set of solutions is a vector space with solutions of the differential equation being vectors in the space.●

Frequently, we encounter a subset W of vectors in a vector space V. By itself, W might be a vector space, but it also might not be one. When a nonempty subset W of a vector space V is itself a vector space, it is called a **subspace** of V. One way to determine whether W is a subspace is to check the requirements of Definition 1.1. But, because vectors in W are vectors in V, most of the properties in Definition 1.1 are inherited by these vectors. According to the following theorem, all that we need check is that W is closed under vector addition and scalar multiplication.

Theorem 1.1 A nonempty subset W of a vector space V is a subspace of V if it is closed under vector addition and scalar multiplication; that is, for any two vectors $\mathbf{u}$ and $\mathbf{v}$ and any scalar a, the vectors $\mathbf{u}+\mathbf{v}$ and $a\mathbf{u}$ are in W.

Proof Since vectors in W are in V, they must satisfy properties (1),(2),(5),(6),(7), and (8) in Definition 1.1. Since W is closed under vector addition and scalar multiplication, it follows that if $\mathbf{v}$ is in W so also is $-\mathbf{v}$ and $\mathbf{v}-\mathbf{v}=\mathbf{0}$, properties (3) and (4).∎

Below are subsets of vector spaces; some are subspaces, some are not.

(1) There are many $\mathcal{G}^2$ subspaces of $\mathcal{G}^3$. For instance, dropping the third component v_z of vectors $\langle v_x, v_y, x_z\rangle$ in $\mathcal{G}^3$ leads to a subspace; vectors $\langle v_x, v_y\rangle$ in the xy-plane. Requiring components to satisfy $3v_x+2v_y-v_z=0$ defines another subspace; vectors whose tips lie in the plane $3x+2y-z=0$. The addition of any two vectors in the plane, or any multiple of a vector in the plane, is also in the plane. Requiring components to satisfy $3v_x+2v_y-v_z=5$, a nonhomogeneous version of the previous condition, does not lead to a subspace. Vectors with tips in this plane do not add to give a vector with its tip in the plane, and a multiple of a vector with tip in the plane does not have its tip in the plane.

(2) Corresponding to the situation in (1), the set of vectors (x_1, x_2, x_3) in $\mathcal{R}^3$ that satisfies $x_1+2x_2+4x_3=0$ constitutes a subspace. Vectors that satisfy $x_1+2x_2+4x_3=1$ do not form a subspace.

(3) The set of all real, diagonal $n\times n$ matrices is a subspace of $M_{n,n}(\mathcal{R})$. It is also a subspace of $M_{n,n}(\mathcal{C})$.

(4) The space $C^n[a,b]$ of real-valued functions that have continuous n^{th} derivatives on the interval $[a,b]$ is a subspace of $C^{n-1}[a,b]$ which is a subspace of $C^{n-2}[a,b]$, which is a subspace of $C^{n-3}[a,b]$, and so on until $C^0[a,b]$.

Since a subspace of a vector space is itself a vector space, it must contain the zero vector. We state this as a corollary to Theorem 1.1.

Corollary 1.1.1 Every subspace of a vector space contains the zero vector.

The only reason that we state this is that it is sometimes a quick way to prove that a subset of a vector space is not a subspace. If the subset does not contain the zero vector, it cannot be a subspace. Here is an example.

Example 1.2 Is the subset of all $m\times n$ real matrices with $(1,1)$ entry equal to 1 a subspace of $M_{m,n}(R)$?

Solution Since the subset does not contain the $m\times n$ zero matrix, the subset is not a subspace.●

Example 1.3 The equation $Ax+By+Cz=0$, where A, B, and C are constants, describes a plane through the origin in space. Show that the subset of all vectors that lie in this plane constitutes a subspace of $\mathcal{G}^3$. Describe the subspace algebraically.

Solution First, recall that we have agreed to draw vectors in $\mathcal{G}^3$ with their tails at the origin. Since the plane passes through the origin, it follows that the sum of any two vectors in the plane is a vector in the plane, and a constant times any vector in the plane is also in the plane. Hence, the subset is a subspace. The components $\langle v_x, v_y, v_z\rangle$ of any vector in the subspace must satisfy $Av_x + Bv_y + Cv_z = 0$.•

Example 1.4 The equations $A_1x+B_1y+C_1z = 0$ and $A_2x+B_2y+C_2z = 0$, where all six coefficients are constants, describe a line through the origin in space. Show that the subset of all vectors that lie along this line constitutes a subspace of $\mathcal{G}^3$. Describe the subspace algebraically.

Solution Because the line passes through the origin, the sum of any two vectors along the line is a vector along the line, and a constant times any vector along the line is also along the line. Hence, the subset is a subspace. The components $\langle v_x, v_y, v_z\rangle$ of any vector in the subspace must satisfy $A_1v_x + B_1v_y + C_1v_z = 0$, $A_2v_x + B_2v_y + C_2v_z = 0$.•

Example 1.5 Is the subset of vectors $\langle v_1, v_2, \ldots, v_n\rangle$ in $\mathcal{G}^n$ that satisfy the equation $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$, where the a_i are constants, not all zero, a subspace?

Solution Since the set is closed under vector addition and scalar multiplication, the set is a subspace. It is called a hyperplane in $\mathcal{G}^n$.•

The smallest subspace of a vector space V is the subspace consisting only of the zero vector in V; the largest subspace is V itself. These may be the only two subspaces of V. If there are other subspaces, they are called **proper subspaces**. Every proper subspace must contain the zero vector and at least one nonzero vector, and there must be at least one vector in the space that is not in the subspace.

EXERCISES 1.2

In Exercises 1–11 determine whether the set of vectors constitutes a vector space. If the set is not a vector space, find at least one of the properties in the definition that fails to be met.

1. For any fixed values of m and n, the set $M_{m,n}(\mathcal{R})$ of all $m \times n$ real matrices with all real numbers $\mathcal{R}$ as scalars.

2. For any fixed values of m and n, the set $M_{m,n}(\mathcal{C})$ of all $m \times n$ complex matrices with all complex numbers $\mathcal{C}$ as scalars.

3. The set of all real polynomials $P_n(x)$ of degree less than or equal to a fixed integer $n \geq 0$ with scalars $\mathcal{R}$.

4. The set of all complex polynomials $P_n(z)$ of degree less than or equal to a fixed integer $n \geq 0$ with scalars $\mathcal{C}$.

5. The set of all pairs (x_1, x_2) of real numbers with addition and scalar multiplication defined as

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2), \quad a(x_1, x_2) = (ax_1, ax_2).$$

6. The set of triples (x_1, x_2, x_3) of real numbers with addition and scalar multiplication defined as

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1 + 1, x_2 + y_2 + 1, x_3 + y_3 + 1), \quad a(x_1, x_2, x_3) = (ax_1, ax_2, ax_3).$$

7. (a) The set of solutions of the differential equation

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0.$$

(b) The set of solutions of the differential equation

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 25.$$

(c) Explain the difference between the two situations.

8. (a) The set of solutions of the differential equation

$$\frac{dy}{dx} = y.$$

(b) The set of solutions of the differential equation

$$\frac{dy}{dx} = y^2.$$

(c) Explain the difference between the two situations.

9. The set of 2×2 real matrices with positive determinants.

10. The set of 2×2 real matrices with nonnegative determinants.

11. The set of all polynomials of degree 3.

12. Determine whether the set of infinite, real sequences $(c_1, c_2, \ldots)$ with addition and scalar multiplication defined as usual,

$$(c_1, c_2, \ldots) + (d_1, d_2, \ldots) = (c_1 + d_1, c_2 + d_2, \ldots), \qquad k(c_1, c_2, \ldots) = (kc_1, kc_2, \ldots)$$

is a vector space.

13. If we demand that sequences in Exercise 12 be convergent, is the set still a vector space?

14. If we demand that sequences in Exercise 12 are such that the series $\sum_{n=1}^{\infty} c_n$ is convergent, is the set still a vector space?

15. If we demand that sequences in Exercise 12 are such that the series $\sum_{n=1}^{\infty} c_n^2$ is convergent, is the set still a vector space?

16. In Exercises 14 and 15, which space is a subspace of the other?

In Exercises 17–37 determine whether the subset of vectors constitutes a subspace.

17. The subset in $\mathcal{R}^3$ of vectors of the form $(x_1, x_2, 2x_1 - 3x_2)$.

18. The subset in $\mathcal{R}^3$ of vectors of the form $(x_1, x_2, x_1^2 + x_2)$.

19. If $\mathbf{v}$ is the vector $\langle 1, 2, 3 \rangle$, the subset of vectors in $\mathcal{G}^3$ of the form $a\mathbf{v}$, where $a \geq 0$.

20. (a) The subset of vectors in $\mathcal{G}^3$ whose x-, y-, and z-components v_x, v_y, and v_z satisfy the equation $5v_x - 2v_y + 3v_z = 0$. Does your verification depend on the coefficients 5, -2, and 3, or would it be true for any values of the coefficients?

(b) The subset of vectors in $\mathcal{G}^3$ whose x-, y-, and z-components v_x, v_y, and v_z satisfy the equation $5v_x - 2v_y + 3v_z = 7$. Does your verification depend on the number 7, or would it be valid for any nonzero constant?

(c) Determine whether what you have shown in parts (a) and (b) is equivalent to saying the following in $\mathcal{G}^3$: Vectors whose tips line in a plane constitute a subspace if, and only if, the plane passes through the origin.

21. (a) The subset of vectors in $\mathcal{G}^3$ whose x-, y-, and z-components v_x, v_y, and v_z satisfy the equations $5v_x - 4v_y + v_z = 0$ and $v_x - 2v_y = 0$. Does your verification depend on the coefficients in the equations or is it true for any values of the coefficients?

(b) The subset of vectors in $\mathcal{G}^3$ whose x-, y-, and z-components v_x, v_y, and v_z satisfy the equations $5v_x - 4v_y + v_z = 2$ and $v_x - 2v_y = 1$. Does your verification depend on the numbers 1 and 2, or would it be valid for any nonzero constants?

(c) Determine whether what you have shown in parts (a) and (b) is equivalent to saying the following in $\mathcal{G}^3$: Vectors whose tips line on a line constitute a subspace if, and only if, the line passes through the origin.

22. The subset in $C^0(a,b)$ of all even continuous functions.

23. The subset in $C^0(a,b)$ of all odd continuous functions.

24. The subset $C^n(a,b)$ in $C^0(a,b)$.

25. The subset in $C^0(-\infty,\infty)$ of polynomials $P(x)$.

26. The subset in $P_n(x)$ consisting of all polynomials whose value at $x=0$ is 0.

27. The subset in $P_n(x)$ consisting of all polynomials whose value at $x=0$ is 5. Does your verification depend on the number 5, or would it be valid for any nonzero constant?

28. The subset in $P_n(x)$ consisting of all polynomials whose value at $x=1$ is 0. Does your verification depend on the number 1, or would it be valid for any nonzero constant?

29. The subset in $P_n(x)$ consisting of all polynomials whose value at $x=1$ is 5. Does your verification depend on the numbers 1 and 5, or would it be valid for any nonzero constants?

30. The subset of all functions with convergent Maclaurin series on the interval $(-1,1)$ in the space $C^\infty[-1,1]$?

31. The subset in $M_{m,n}(\mathcal{R})$ of matrices whose entries are all greater than or equal to zero.

32. The subset in $M_{n,n}(\mathcal{R})$ of symmetric matrices.

33. The subset in $M_{n,n}(\mathcal{R})$ of matrices that are not symmetric.

34. The subset in $M_{2,2}(\mathcal{R})$ of matrices for which the sum of the four entries is equal to 5.

35. The subset in $M_{2,2}(\mathcal{R})$ of matrices of the form $\begin{pmatrix} a & 1 \\ b & c \end{pmatrix}$.

36. (a) The subset of solutions of the form $y(x)=ce^{2x}$, where c is a constant, of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

(b) The subset of solutions of the form $y(x)=ce^{2x}+1$, where c is a constant, of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2.$$

(c) Explain the difference between the situations in parts (a) and (b).

37. The subset in $P_2(x,y)$ of polynomials of the form $p(x,y)=axy+bx^2+c$, where a, b, and c are constants.

38. Is the subset of all real arithmetic sequences a subspace of the vector space in Exercise 12? Is it a subspace of the space in Exercise 13?

39. Is the subset of all real geometric sequences a subspace of the vector space in Exercise 13?

40. The **intersection** $W_1 \cap W_2$ of two subspaces W_1 and W_2 of a vector space V is the set of all vectors in V that are in both W_1 and W_2. Prove that the intersection is a subspace of V. Is it also a subspace of W_1 and of W_2?

41. The **union** $W_1 \cup W_2$ of two subspaces W_1 and W_2 of a vector space V is the set of all vectors in V that are in either W_1 or W_2. Is the union a subspace of V?

42. The **sum** $W_1 + W_2$ of two subspaces of a vector space V is the set of all vectors $\mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in W_1 and $\mathbf{w}_2$ is in W_2. Prove that the sum is a subspace of V.

43. Prove that the union $W_1 \cup W_2$ of two subspaces W_1 and W_2 of a vector space is a subspace if, and only if, W_1 is a subspace of W_2, or W_2 is a subspace of W_1.

44. Let V be the vector space of all functions defined on the interval $0 \leq x \leq 1$, and let V_1, V_2, V_3, and V_4 be the following subspaces:

$$\begin{aligned} V_1 &= \text{subspace of all polynomials in } x, \\ V_2 &= \text{subspace of all differentiable functions on } 0 \leq x \leq 1, \\ V_3 &= \text{subspace of all continuous functions on } 0 \leq x \leq 1, \\ V_4 &= \text{subspace of all integrable functions on } 0 \leq x \leq 1. \end{aligned}$$

Show that V_i is a subspace of V_j for $i \leq j$.

45. Show that the only proper subspaces of $\mathcal{G}^2$ are lines through the origin.

46. Show that the only proper subspaces of $\mathcal{G}^3$ are lines and planes through the origin.

47. Is the subset of vectors $(x_1, x_2, \ldots, x_n)$ in $\mathcal{R}^n$ that satisfy m homogeneous, linear, equations

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \quad i = 1, \ldots, m,$$

a subspace of $\mathcal{R}^n$?

48. Let V be the set of all straight lines through the origin in the xy-plane. If L_1 and L_2 are two lines in V, define $L_1 + L_2$ to be the line through the origin with slope equal to the sum of the slopes of L_1 and L_2. If a is a real scalar, define aL_1 to be the line through the origin with slope equal to a times the slope of L_1. Is V a real vector space?

49. Let V be the set of all circles in the xy-plane with centres at the origin. Include in V a circle with centre at the origin and radius zero. If C_1 and C_2 are two circles in V, define $C_1 + C_2$ to be the circle with centre at the origin with radius equal to the sum of the radii of C_1 and C_2. If a is a real scalar, define aC_1 to be the circle with centre at the origin and radius equal to $|a|$ times the radius of C_1. Is V a real vector space?

50. Find a subset in $\mathcal{G}^2$ that is closed under vector addition, but not under scalar multiplication.

51. Find a subset in $\mathcal{G}^2$ that is closed scalar multiplication, but not under vector addition.

52. Find a subset of $\mathcal{G}^3$ that contains the zero vector, but it is not a subspace.

Answers

1. Yes **2.** Yes **3.** Yes **4.** Yes
5. No; violates property (1) that vector addition must be commutative
6. No; violates property (6)
7. Yes (b) No; violates closure under vector addition (c) One equation is homogeneous and the other is not.
8.(a) Yes (b) No; violates closure under vector addition (c) One equation is linear and the other is not.
9. No; violates property (3) **10.** No; violates closure under vector addition
11. No; violates closure under vector addition **12.** Yes **13.** Yes **14.** Yes **15.** Yes **16.** Neither
17. Yes **18.** No **19.** No **20.**(a) Yes; does not depend (b) No; does not depend (c) Yes
21.(a) Yes; does not depend (b) No; does not depend (c) Yes **22.** Yes **23.** Yes **24.** Yes **25.** Yes
26. Yes **27.** No; does not depend **28.** Yes; does not depend **29.** No; does not depend **30.** Yes
31. No **32.** Yes **33.** No **34.** No **35.** No
36.(a) Yes (b) No (c) One equation is homogeneous and the other is not. **37.** Yes **38.** Yes, No
39. No **40.** Yes **41.** Sometimes **47.** Yes **48.** Yes **49.** No
50. All vectors of the form $a\langle 1, 2\rangle$, where a is a positive constant.
51. All vectors of the form $a\langle 1, 2\rangle$ or $a\langle 3, 4\rangle$, where a is a real constant.
52. The subset in Exercise 19.

§1.3 Linearly Independent and Linearly Dependent Sets of Vectors

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a set of m vectors in a vector space V, and $c_1, c_2, \ldots, c_m$ are constants, we say that the vector

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \tag{1.3}$$

is a **linear combination** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. For example, in $\mathcal{G}^2$, every vector $\mathbf{v}$ can be expressed as a linear combination of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$, where v_x and v_y are the x- and y-components of $\mathbf{v}$. Every vector in $\mathcal{G}^2$ can also be expressed as a linear combination of the vectors $3\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$. To show this, suppose that $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$ is any vector in $\mathcal{G}^2$, and consider finding constants c_1 and c_2 so that

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} = c_1(3\hat{\mathbf{i}} + \hat{\mathbf{j}}) + c_2(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) = (3c_1 - c_2)\hat{\mathbf{i}} + (c_1 + 2c_2)\hat{\mathbf{j}}.$$

When we equate x- and y-components, we obtain

$$v_x = 3c_1 - c_2, \quad v_y = c_1 + 2c_2,$$

two nonhomogeneous, linear equations in c_1 and c_2. The solution is $c_1 = (2v_x + v_y)/7$ and $c_2 = (3v_y - v_x)/7$; that is,

$$\mathbf{v} = \frac{1}{7}(2v_x + v_y)(3\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \frac{1}{7}(3v_y - v_x)(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}).$$

For example, if $\mathbf{v} = 10\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$, then, its representation in terms of $3\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ is

$$10\hat{\mathbf{i}} - 3\hat{\mathbf{j}} = \frac{17}{7}(3\hat{\mathbf{i}} + \hat{\mathbf{j}}) - \frac{19}{7}(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}).$$

Some vectors in $\mathcal{G}^2$ can be expressed in terms of the vectors $\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$, but not all of them. This is clear geometrically. The two vectors are parallel, and therefore another vector in the same direction as these can be expressed in terms of them in infinitely many ways, but a vector in a different direction cannot be expressed in terms of them. To see this algebraically, consider finding constants c_1 and c_2 so that

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} = c_1(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) + c_2(2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) = (c_1 + 2c_2)\hat{\mathbf{i}} + (2c_1 + 4c_2)\hat{\mathbf{j}}.$$

When we equate x- and y-components, we obtain

$$v_x = c_1 + 2c_2, \quad v_y = 2c_1 + 4c_2.$$

Because the determinant of the matrix of coefficients of the system is zero, for some choices of v_x and v_y, there will be an infinity of solutions, and for other choices, there will be no solution.

The situation is identical in $\mathcal{R}^2$. Every vector (x_1, x_2) in $\mathcal{R}^2$ can be expressed as a linear combination of the vectors $(1, 0)$ and $(0, 1)$, namely $(x_1, x_2) = x_1(1, 0) + x_2(0, 1)$. Every vector in $\mathcal{R}^2$ can also be expressed as a linear combination of the vectors $(3, 1)$ and $(-1, 2)$. In particular, the vector $(10, -3)$ can be expressed as $(17/7)(3, 1) - (19/7)(-1, 2)$. Some vectors can be expressed as a linear combination of $(1, 2)$ and $2, 4)$, but not all of them.

We have a similar situation in $\mathcal{G}^3$ and $\mathcal{R}^3$. Every vector in $\mathcal{R}^3$ can be expressed in terms of $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Every vector can also be expressed as linear combinations of the vectors $(1, -2, -4)$, $(-2, 1, 5)$, and $(1, 0, 4)$. For instance, for the vector $(4, -1, 5)$, we would set

$$\begin{aligned}(4, -1, 5) &= c_1(1, -2, -4) + c_2(-2, 1, 5) + c_3(1, 0, 4)\\ &= (c_1 - 2c_2 + c_3, -2c_1 + c_2, -4c_1 + 5c_2 + 4c_3).\end{aligned}$$

When we equate components, we obtain the nonhomogeneous linear equations

$$c_1 - 2c_2 + c_3 = 4, \quad -2c_1 + c_2 = -1, \quad -4c_1 + 5c_2 + 4c_3 = 5.$$

The solution is $c_1 = 1/9$, $c_2 = -7/9$, and $c_3 = 7/3$. Thus,

$$(4, -1, 5) = \frac{1}{9}(1, -2, -4) - \frac{7}{9}(-2, 1, 5) + \frac{7}{3}(1, 0, 4).$$

On the other hand, the vector $(4, -1, 5)$ cannot be expressed as a linear combination of the vectors $(1, -2, -4)$, $(-2, 1, 5)$, and $(-1, -1, 1)$. We leave the reader to verify this.

To determine whether it is possible to express all vectors in a vector space in terms of a given set of vectors, and whether all vectors in the set are necessary, it is useful to define *linearly dependent* and *linearly independent* sets of vectors.

Definition 1.2 A set of nonzero vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is said to be **linearly dependent** if there exists m scalars, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}. \tag{1.4}$$

If the only way a linear combinations of the vectors can be equal to the zero vector is for all coefficients to be zero, the set of vectors is said to be **linearly independent**.

We choose the set to consist of nonzero vectors because if the set contains the zero vector, it is automatically linearly dependent. For instance, if $\mathbf{v}_1$ is the zero vector, then with c_1 any nonzero number, and the remaining constants equal to zero, equation 1.4 is satisfied.

Although linear dependence and linear independence is a property of a set of vectors, we often omit the word "set", and say that the vectors themselves are linearly dependent or linearly independent. Realize, however, that linear dependence or independence is a property of the collection of vectors, not the individual vectors. The vectors $(2, 4)$ and $(-1, -2)$ in $\mathcal{R}^2$ are linearly dependent because

$$(2, 4) + 2(-1, -2) = \mathbf{0}.$$

To find out whether the vectors $(2, 4)$ and $(3, -5)$ are linearly dependent, we determine whether there are constants, not both zero, such that

$$c_1(2, 4) + c_2(3, -5) = \mathbf{0}.$$

By adding the vectors on the left and equating its components to zero, we obtain

$$2c_1 + 3c_2 = 0, \quad 4c_1 - 5c_2 = 0.$$

Since the only solution of this homogeneous system of equations is $c_1 = c_2 = 0$, the vectors are linearly independent.

Example 1.6 Determine whether the vectors $\mathbf{v}_1 = (1, -2, 3)$, $\mathbf{v}_2 = (4, -2, 5)$, and $\mathbf{v}_3 = (-13, 2, -11)$ in $\mathcal{R}^3$ are linearly independent or linearly dependent.

Solution We determine whether there are constants, not all zero, such that

$$\begin{aligned}\mathbf{0} &= c_1(1, -2, 3) + c_2(4, -2, 5) + c_3(-13, 2, -11)\\ &= (c_1 + 4c_2 - 13c_3, -2c_1 - 2c_2 + 2c_3, 3c_1 + 5c_2 - 11c_3).\end{aligned}$$

When we equate components, we obtain three linear, homogeneous equations

$$c_1 + 4c_2 - 13c_3 = 0, \quad -2c_1 - 2c_2 + 2c_3 = 0, \quad 3c_1 + 5c_2 - 11c_3 = 0.$$

Gaussian or Gauss-Jordan elimination shows that there is an infinite number of solutions that can be represented in the form

$$c_1 = -3c_3, \quad c_2 = 4c_3, \quad \text{where } c_3 \text{ is arbitrary.}$$

For instance, if we set $c_3 = 1$, then $c_1 = -3$ and $c_2 = 4$, and

$$\mathbf{0} = -3(1,-2,3) + 4(4,-2,5) + (-13,2,-11).$$

The vectors are therefore linearly dependent.●

The following theorem provides an easier way to think about linear dependence.

Theorem 1.2 A set of nonzero vectors is linearly dependent if, and only if, at least one of the vectors is a linear combination of the other vectors. Alternatively, a set of nonzero vectors is linearly independent if, and only if, none of the vectors is a linear combination of the others.

Proof: If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is linearly dependent, then there exist constants, not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}.$$

If $c_i \neq 0$ for some i, then we can solve the equation for $\mathbf{v}_i$,

$$\mathbf{v}_i = -\frac{c_1}{c_i}\mathbf{v}_1 - \cdots - \frac{c_{i-1}}{c_i}\mathbf{v}_{i-1} - \frac{c_{i+1}}{c_i}\mathbf{v}_{i+1} - \cdots - \frac{c_m}{c_i}\mathbf{v}_m.$$

This shows that $\mathbf{v}_i$ is a linear combinations of the other vectors. Conversely, suppose that one of the vectors, say $\mathbf{v}_i$, is a linear combination of the other vectors,

$$\mathbf{v}_i = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{i-1}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_{i+1} + \cdots + c_m\mathbf{v}_m.$$

We can then write

$$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + c_{i+1}\mathbf{v}_{i+1} + \cdots + c_m\mathbf{v}_m.$$

Since not all coefficients are zero, the vectors are linearly dependent.∎

This theorem does not say that every vector in a linearly dependent set can be expressed in terms of the other vectors in the set; it says that at least one of the vectors can be so represented, but not necessarily all of them. For instance, the set of vectors $\{\hat{\mathbf{i}}, 2\hat{\mathbf{i}}, \hat{\mathbf{k}}\}$ is linearly dependent. The first and second vectors can be expressed in terms of the other two vectors, but $\hat{\mathbf{k}}$ cannot be expresed in terms of $\hat{\mathbf{i}}$ and $2\hat{\mathbf{i}}$.

The following two theorems describe linear dependence for vectors in $\mathcal{G}^2$ and $\mathcal{G}^3$ geometrically. Corollaries gives the equivalent in $\mathcal{R}^2$ and $\mathcal{R}^3$.

Theorem 1.3 Two nonzero vectors in $\mathcal{G}^2$ are linearly dependent if, and only if, they are parallel.

Corollary 1.3.1 Two nonzero vectors in $\mathcal{R}^2$ are linearly dependent if, and only if, each is a multiple of the other.

Theorem 1.4 Three nonzero vectors in $\mathcal{G}^3$ are linearly dependent if, and only if, all three vectors are parallel, or failing this, one of the vectors lies in the plane determined by the other two.

Corollary 1.4.1 Three nonzero vectors in $\mathcal{R}^3$ are linearly dependent if, and only if, all three vectors are multiples of each other, or failing this, one of the vectors is a linear combination of the other two.

The following results pertain to vectors in $\mathcal{G}^n$ or $\mathcal{R}^n$.

Theorem 1.5 If $m > n$, a set of m vectors in $\mathcal{G}^n$ or $\mathcal{R}^n$ is always linearly dependent. (In short, if you have more vectors than components, then the vectors are linearly dependent.)

Proof We will use the parentheses notation of $\mathcal{R}^n$, but it is equally valid with angle-bracket notation of $\mathcal{G}^n$. Let $\mathbf{v}_i = (v_{i1}, v_{i2}, \dots, v_{in})$, $i = 1, \dots, m$ be m vectors in $\mathcal{R}^n$, and consider finding m constants so that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}.$$

When we substitute the components for the vectors and equate components of the left side to zero, we obtain n homogeneous, linear equations in the m constants $c_1, \ldots, c_m$,

$$\begin{aligned} v_{11}c_1 + v_{21}c_2 + \cdots + v_{m1}c_m &= 0, \\ v_{12}c_1 + v_{22}c_2 + \cdots + v_{m2}c_m &= 0, \\ \vdots \qquad\qquad & \quad \vdots \\ v_{1n}c_1 + v_{2n}c_2 + \cdots + v_{mn}c_m &= 0. \end{aligned}$$

When there is more unknowns than equations in a homogeneous, linear system, there is always an infinity of nontrivial solutions. Hence, the vectors are linearly dependent.∎

Theorem 1.6 A set of n nonzero vectors in $\mathcal{G}^n$ or $\mathcal{R}^n$ is linearly dependent if, and only if, the $n \times n$ determinant whose columns (or rows) are the components of the vectors has value 0.

Proof With the notation of Theorem 1.5, it is a question of whether the system of n equations in now n unknown coefficients has a nontrivial solution. But for a system of n homogeneous, linear equations in n unknowns, there are nontrivial solutions if, and only if, the determinant of the coefficient matrix has value zero. But this matrix contains the components of the vectors in its columns, and the proof is complete.∎

Example 1.7 Use Theorem 1.6 to solve Example 1.6.

Solution We evaluate the determinant whose columns are the components of the vectors

$$\det \begin{pmatrix} 1 & 4 & -13 \\ -2 & -2 & 2 \\ 3 & 5 & -11 \end{pmatrix} = 0.$$

Hence, the vectors are linearly dependent.•

Theorem 1.5 describes the situation in $\mathcal{G}^n$ and $\mathcal{R}^n$ when there are more vectors than components. Theorem 1.6 describes the situation when the number of components is the same as the number of vectors. No general statements can be made when there are fewer vectors than components; the vectors might be dependent or independent. To decide, we return to Definition 1.2 or Theorem 1.2. Here are two examples to illustrate.

Example 1.8 Determine whether the vectors $(1, -3, 5, -2)$, $(2, 0, 3, 1)$, and $(-4, -6, 1, -7)$ in $\mathcal{R}^4$ are linearly independent or dependent.

Solution Consider finding constants so that

$$c_1(1, -3, 5, -2) + c_2(2, 0, 3, 1) + c_3(-4, -6, 1, -7) = \mathbf{0}.$$

When we equate components, we obtain the equations

$$c_1 + 2c_2 - 4c_3 = 0, \quad -3c_1 - 6c_3 = 0, \quad 5c_1 + 3c_2 + c_3 = 0, \quad -2c_1 + c_2 - 7c_3 = 0.$$

There is an infinity of solutions of this system, representable in the form

$$c_1 = -2c_3, \quad c_2 = 3c_3.$$

This implies that the vectors are linearly dependent.•

Example 1.9 Determine whether the vectors $(1, -3, 5, -2)$, $(2, 0, 3, 1)$, and $(-4, -6, 1, -5)$ in $\mathcal{R}^4$ are linearly independent or dependent.

Solution Consider finding constants so that

$$c_1(1, -3, 5, -2) + c_2(2, 0, 3, 1) + c_3(-4, -6, 1, -5) = \mathbf{0}.$$

When we equate components, we obtain the equations

$$c_1 + 2c_2 - 4c_3 = 0, \quad -3c_1 - 6c_3 = 0, \quad 5c_1 + 3c_2 + c_3 = 0, \quad -2c_1 + c_2 - 5c_3 = 0.$$

The only solution of this system is $c_1 = c_2 = c_3 = 0$. The vectors are therefore linearly independent.●

The following examples consider linear independence and dependence in more abstract vector spaces. Because vectors in these spaces do not yet have components, we must return to Definition 1.2 or Theorem 1.2.

Example 1.10 Determine whether the polynomials (vectors) $4x$, $3x^2$, and $-2x^2 + 5x$ in $P_2(x)$ are linearly independent or dependent.

Solution Since $-2x^2 + 5x$ is a linear combination of $4x$ and $3x^2$, namely,

$$-2x^2 + 5x = -\frac{2}{3}(3x^2) + \frac{5}{4}(4x),$$

the polynomials are linearly dependent.●

Example 1.11 The functions e^x, xe^x, and x^2e^x are vectors in the vector space of solutions of the linear, homogeneous differential equation

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0.$$

Are they linearly dependent or independent?

Solution Consider finding constants c_1, c_2, and c_3 so that

$$0 = c_1(e^x) + c_2(xe^x) + c_3(x^2e^x) = (c_1 + c_2x + c_3x^2)e^x.$$

Since this must be true for all x, we set $x = 0$, in which case, $c_1 = 0$. If we remove the e^x from the equation, and then differentiate the result, we obtain $c_2 + 2c_3x = 0$. Setting $x = 0$ now gives $c_2 = 0$. It then follows that $c_3 = 0$ also. Thus, the solutions (vectors) are linearly independent. Alternatively, it is reasonably clear that none of the vectors is a linear combination of the other two, and therefore the vectors are linearly independent.●

Example 1.12 Is the set of vectors $\{1 + x, 2 + 3x - 2x^2, 5x - 4x^2\}$ linearly dependent or independent in $P_2(x)$?

Solution There are two approaches that we could take. A third will present itself in the next section. First, we could take the approach of Example 1.11, and consider finding constants c_1, c_2, and c_3 so that

$$0 = c_1(1 + x) + c_2(2 + 3x - 2x^2) + c_3(5x - 4x^2).$$

Since this is to be true for all x, we set $x = 0$, in which case $0 = c_1 + 2c_2$. If we set $x = 1$, we get a second equation, $0 = 2c_1 + 3c_2 + c_3$. Finally, if we set $x = -1$, we get $0 = -3c_2 - 9c_3$. The only solution of these equations is $c_1 = c_2 = c_3 = 0$, so that the vectors are linearly independent. Instead of substituting values of x, suppose that we rearrange terms in the equation,

$$0 = (c_1 + 2c_2) + (c_1 + 3c_2 + 5c_3)x + (-2c_2 - 4c_3)x^2.$$

Since the vectors 1, x, and x^2 are linearly independent (see Exercise 17), it follows that

$$0 = c_1 + 2c_2, \quad 0 = c_1 + 3c_2 + 5c_3, \quad 0 = -2c_2 - 4c_3.$$

Once again, the only solution is the trivial one.●

EXERCISES 1.3

In Exercises 1–12 determine whether the set of vectors is linearly dependent or linearly independent.

1. $\{\langle 1, 2\rangle, \langle 3, 5\rangle\}$

2. $\{(2, -1), (-4, 2)\}$

3. $\{(1, 3), (2, -3), (4, 10)\}$

4. $\{\langle 1, -2, 4\rangle, \langle 2, -5, 0\rangle\}$

5. $\{\langle 2, -4, 1\rangle, \langle 1, -3, 5\rangle, \langle 5, -11, 7\rangle\}$

6. $\{\langle 3, 2, -1\rangle, \langle 3, 5, 8\rangle, \langle -2, 4, 1\rangle\}$

7. $\{\langle 3, 2, -1\rangle, \langle 3, 5, 8\rangle, \langle -2, 4, 1\rangle, \langle 1, 1, 1\rangle\}$

8. $\{(1, -1, 1, -1), (1, 3, -2, 5), (4, 0, 2, 5)\}$

9. $\{(2, 0, 3, 4), (1, -3, 5, 1), (1, 0, 0, 3), (4, -3, 8, 8)\}$

10. $\{\langle 2, 0, 3, 4\rangle, \langle 1, 1, 1, 2\rangle, \langle -2, 4, 1, 3\rangle, \langle 4, 3, 2, 1\rangle\}$

11. $\{(3 - i, 2, -i), (1, 0, 1), (i, -i, 3)\}$

12. $\{(1, -1, -i), (i, 1, 1 - i), (1 + 2i, -3i, 1 - i)\}$

13. Show that the vectors $\mathbf{u} = \langle 2, -1, 4\rangle$, $\mathbf{v} = \langle 3, 5, -2\rangle$, and $\mathbf{w} = \langle 1, 1, 0\rangle$ are linearly independent. Express the vector $\langle 3, 5, 8\rangle$ as a linear combination of these vectors.

14. (a) Show that the vectors $\langle 2, -1, 4\rangle$, $\langle 3, 5, -2\rangle$, and $\langle 1, -7, 10\rangle$ are linearly dependent.
(b) Show that the vector $\langle 3, 5, 8\rangle$ cannot be expressed as a linear combination of these vectors.
(c) Show that the vector $\langle -4, -11, 8\rangle$ can be expressed as a linear combination of these vectors.

15. Suppose the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly dependent. Does it follow that $\mathbf{v}_1$ can be expressed in terms of $\mathbf{v}_2$, …, $\mathbf{v}_m$? Explain.

16. (a) Prove that the functions e^x, e^{-2x}, and xe^{-2x} are solutions of the differential equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - 4y = 0.$$

(b) Is the set of them linearly independent?

17. Show that the set of vectors $\{1, x, x^2, \ldots, x^n\}$ in $P_n(x)$ is linearly independent.

18. In the space of 2×2 real, symmetric matrices, are the following matrices linearly dependent or independent?

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

19. Prove that every subset of a set of linearly independent vectors is also linearly independent. Is the same result true for linearly dependent sets of vectors?

20. (a) Are the vectors $\mathbf{v}_1 = (-2, 1, 1)$, $\mathbf{v}_2 = (1, -2, 1)$, and $\mathbf{v}_3 = (1, 1, -2)$ linearly dependent in $\mathcal{R}^3$?
(b) Are $\mathbf{v}_1$ and $\mathbf{v}_2$ linearly dependent?
(c) Are $\mathbf{v}_2$ and $\mathbf{v}_3$ linearly dependent?
(d) Are $\mathbf{v}_1$ and $\mathbf{v}_3$ linearly dependent?

21. Repeat Exercise 20 for the vectors $\mathbf{v}_1 = (-2, 1, 1)$, $\mathbf{v}_2 = (4, 2, -2)$, and $\mathbf{v}_3 = (5, 1, -2)$

22. Prove or disprove the following statement: If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, where $m \geq 4$ is linearly dependent, then given any two vectors in the set, then their sum can be expressed as a linear combination of the other vectors in the set.

23. The first four Legendre polynomials are

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \frac{3x^2 - 1}{2}, \quad p_3(x) = \frac{5x^3 - 3x}{2}.$$

Are they linearly independent in the space $P_3(x)$?

24. Determine whether the set of polynomials $\{1 + 2x - x^2, 3 + 4x - x^3, 2x + 4x^2 - x^3, 5 + 10 + 2x^2 - 2x^3\}$ is linearly independent or dependent.

25. (a) Is the pair of functions $\{2x, |x|\}$ linearly independent or dependent in the space $C^0[-a, a]$?
(b) Is the pair independent or dependent in $C^0[0, a]$?

26. Find all values of θ in order that the vectors (functions) $\sin(x+\theta)$ and $\sin x$ be linearly dependent in $C^0[-\pi,\pi]$.

27. Show that in $P_n(x)$, any set of n polynomials, one of degree 0, one of degree 1, one of degree 2, …, and one of degree n is linearly independent.

28. You are given that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly independent, but the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{w}\}$ is linearly dependent. Show that $\mathbf{w}$ is a linear combination of the $\mathbf{v}_i$.

Answers

1. Linearly independent **2.** Linearly dependent **3.** Linearly dependent
4. Linearly independent **5.** Linearly dependent **6.** Linearly independent
7. Linearly dependent **8.** Linearly independent **9.** Linearly dependent
10. Linearly independent **11.** Linearly independent **12.** Linearly dependent **13.** $10\mathbf{u}+16\mathbf{v}-65\mathbf{w}$
15. No **16.**(b) Yes **18.** Dependent **19.** No **20.** (a) Yes (b) No (c) No (d) No
21. (a) Yes (b) Yes (c) No (d) No **22.** False **23.** Yes **24.** Linearly dependent
25.(a) Linearly independent (b) Linearly dependent **26.** $n\pi$, n an integer

§1.4 Basis and Dimension of a Vector Space

In this section we discuss basis and dimension of a vector space.

Definition 1.3 If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a set of m vectors in a vector space, the set of all linear combinations of the vectors is called the **span** of the vectors.

For instance, in $\mathcal{G}^2$, the span of the vector $\mathbf{v}_1 = \langle 1, -2\rangle$ is all vectors of the form $c_1\langle 1, -2\rangle$, all multiples of $\mathbf{v}_1$. The span of the vectors $\mathbf{v}_1 = \langle 1, -2, 3\rangle$ and $\mathbf{v}_2 = \langle -3, 4, 0\rangle$ in $\mathcal{G}^3$ is all vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\langle 1, -2, 3\rangle + c_2\langle -3, 4, 0\rangle$; it is all vectors in the plane through the origin defined by $\mathbf{v}_1$ and $\mathbf{v}_2$. The span of the vectors $\mathbf{v}_1 = (3, -2, 6)$, $\mathbf{v}_2 = (-2, 5, 1)$, and $\mathbf{v}_3 = (4, 0, 7)$ in $\mathcal{R}^3$ is all linear combinations

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1(3, -2, 6) + c_2(-2, 5, 1) + c_3(4, 0, 7).$$

The functions $1 - 2x$, $2x^2 - 5x^3$, and e^x are vectors in $C^2(-\infty, \infty)$, the space of twice continuously differentiable functions on the interval $-\infty < x < \infty$. Their span is all functions of the form $c_1(1 - 2x) + c_2(2x^2 - 5x^3) + c_3e^x$. The span of the three solutions (vectors) e^x, xe^x and x^2e^x of the differential equation in Example 1.11 is all solutions of the form $c_1e^x + c_2xe^x + c_3x^2e^x$.

Every vector $\mathbf{v}$ in $\mathcal{G}^2$ is a linear combination of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$; coefficients in the linear combination being the components $\langle v_x, v_y\rangle$ of $\mathbf{v}$; that is, $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$. In other words, the span of the vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is the whole space $\mathcal{G}^2$. Similarly, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ span $\mathcal{G}^3$. Correspondingly, the vectors $(1, 0)$ and $(0, 1)$ span $\mathcal{R}^2$ and the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ span $\mathcal{R}^3$. More generally, if $\mathbf{e}_i$ is the vector in $\mathcal{R}^n$ all of whose components are zero except the i^{th} component which is one, then the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ spans $\mathcal{R}^n$.

Theorem 1.7 The span of a (nonempty) set of vectors in a vector space is a subspace.

Proof Since the span is all linear combinations of vectors in the set, the condition of closure under addition and scalar multiplication of Theorem 1.1 is automatically satisfied.∎

Intuitively, we should be able to take enough vectors in a set to span the whole space. We query whether there is a minimum number of vectors that a set must contain in order to span the entire space. We describe such a set in the following definition.

Definition 1.4 When a set of vectors in a vector space V spans the space, and there is not a smaller number of vectors that spans the space, then the set of vectors is said to be a **basis** for the space. The number n of vectors in a basis is called the **dimension** of the space, and the space is said to be n-**dimensional**. The space consisting of only the zero vector is said to be **0-dimensional**. If a basis does not consist of a finite number of vectors, the space is said to be **infinite-dimensional**.

Vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are a basis for $\mathcal{G}^2$, which is therefore 2-dimensional, and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are a basis for the 3-dimensional space $\mathcal{G}^3$. These bases (the plural of basis) are called the **natural** or **standard** bases for $\mathcal{G}^2$ and $\mathcal{G}^3$; they are perhaps the only bases with which most readers are familiar, and they are often the ones in which many problems are initially formulated. Correspondingly, $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ is the natural basis for $\mathcal{R}^3$. The natural basis for $\mathcal{R}^n$ consists of the n vectors $\mathbf{e}_i$ $(i = 1, \ldots, n)$ where all entries in $\mathbf{e}_i$ are zero except the i^{th}, which is one. With every vector space, we can define what we mean by its natural basis, but different choices can be made. Everyone agrees with the above choices for $\mathcal{G}^3$ and $\mathcal{R}^n$. The set of vectors $\{1, x, x^2, x^3\}$ is a basis for the space $P_3(x)$ of real polynomials of degree less than or equal to 3 (they span the space, and no smaller set of vectors can do so). Everyone agrees that this is the natural basis for the space. On the other hand, there is no obvious choice for the space of all solutions to the differential equation $d^2y/dx^2 - k^2y = 0$, where $k > 0$ is a constant. Two possibilities are $\{e^{kx}, e^{-kx}\}$ and $\{\cosh kx, \sinh kx\}$, and neither is more "natural" than the other.

When $\mathbf{v} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$ is a vector in $\mathcal{G}^2$, we call 3 and -4 the components of the vector with respect to the basis $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}\}$, and we have been accustomed to writing these components in the form $\langle 3, -4 \rangle$. Similarly, the components of a vector $\mathbf{v} = 2\mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$ in $\mathcal{R}^3$ are 2, -3, and 4, and we write $\mathbf{v} = (2, -3, 4)$, understanding that these are components of the vector with respect to the natural basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. When $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for an n-dimensional vector space V, then every vector $\mathbf{v}$ in the space can be expressed as a linear combination of the basis vectors,

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n. \tag{1.5}$$

We call coefficients v_i **components of v with respect to the basis** $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}^\dagger$. We often write the components as an n-tuple $(v_1, v_2, \ldots, v_n)$. Although we have the same notation for vectors in $\mathcal{R}^n$, this is not a vector in $\mathcal{R}^n$. In other words, when we see an n-tuple, context must make it clear whether the n-tuple is a vector in $\mathcal{R}^n$ or whether the n-tuple represents the n components of a vector in some n-dimensional space with respect to some basis of the space. Should we wish to denote that the components of a vector are with respect to a particular basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, we sometimes use a subscript as follows $\mathbf{v}_\mathbf{b} = (v_1, v_2, \ldots, v_n)$. We might even want to place the subscript on each component and write $\mathbf{v}_\mathbf{b} = (v_{b1}, v_{b2}, \ldots, v_{bn})$.

Example 1.13 What are the components of the vector $p(x) = 3 - 5x + 2x^3$ in $P_3(x)$ with respect to the natural basis for the space?

Solution Components are $(3, -5, 0, 2)$.●

Example 1.14 Accepting for the moment that the vectors $\{4, 1 + x, 3 - 5x + 2x^2\}$ constitute a basis for $P_2(x)$, find the components of the vector $p(x) = 1 + 3x - 4x^2$ with respect to this basis.

Solution We must express the vector as a linear combination of the basis vectors. We can do this as follows

$$\begin{aligned}
1 + 3x - 4x^2 &= -2(2x^2 - 5x + 3) + (1 + 3x) + (-10x + 6) \\
&= -2(2x^2 - 5x + 3) - 7x + 7 \\
&= -2(2x^2 - 5x + 3) + \frac{7}{5}(3 - 5x) + 7 - \frac{21}{5} \\
&= -2(2x^2 - 5x + 3) + \frac{7}{5}(3 - 5x) + \frac{14}{5} \\
&= -2(2x^2 - 5x + 3) + \frac{7}{5}(3 - 5x) + \frac{7}{10}(4).
\end{aligned}$$

Components of the vector are therefore $(7/10, 7/5, -2)$.●

It is important that the order of the components of a vector be the same as the order of the basis vectors. (We adhered to this in the above example.) In other words, although we shall not do so, it would be more appropriate for us to talk about *ordered basis*, and *ordered components* of vectors. Just remember to list components of vectors in the same order as basis vectors.

Much of our work on vector spaces involves bases and therefore it is important to know when a set of vectors constitutes a basis for the space. For instance, do the vectors $(1, -1)$ and $(-2, 3)$ constitute a basis for $\mathcal{R}^2$, and do any three vectors in $\mathcal{R}^3$ form a basis for the space. The following theorem gives an easy way to determine whether a set of vectors is a basis for a vector space, provided we know the dimension of the space.

Theorem 1.8 In an n-dimensional vector space, any set of n linearly independent vectors constitutes a basis.

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† Many authors refer to these as coordinates of a vector.

Proof: Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of n linearly independent vectors in an n-dimensional vector space V. First we show that the vectors span V. If $\mathbf{v}$ is any vector in V, consider finding constants c_1, c_2, …, c_n so that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

If we denote the components of $\mathbf{v}$ (with respect to some basis) by v_j, and those of $\mathbf{v}_i$ by v_{ij}, then this equation can be written in the form

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = c_1 \begin{pmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{pmatrix} + c_2 \begin{pmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{pmatrix} + \cdots + c_n \begin{pmatrix} v_{n1} \\ v_{n2} \\ \vdots \\ v_{nn} \end{pmatrix},$$

or,

$$\begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

This is a system of n linear, nonhomogeneous equations in the n unknowns c_i. Because the columns of the coefficient matrix are components of the linearly independent vectors $\mathbf{v}_i$, its determinant is nonzero. Hence, there is a unique solution for the c_i, and the $\mathbf{v}_i$ span V. Now we verify that no smaller set of linearly independent vectors can span the space. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for the space, and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is a linearly independent set of vectors that spans the space where $m < n$. Each basis vector $\mathbf{v}_i$ can be expressed in terms of basis vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$,

$$\mathbf{v}_i = c_{i1}\mathbf{w}_1 + \cdots + c_{im}\mathbf{w}_m = \sum_{j=1}^{m} c_{ij}\mathbf{w}_j, \quad i = 1, \ldots, n.$$

Consider finding constants d_i, $i = 1, \ldots, n$ so that

$$\sum_{i=1}^{n} d_i\mathbf{v}_i = \mathbf{0}.$$

If we substitute the expression for $\mathbf{v}_i$ in terms of the $\mathbf{w}_j$, we obtain

$$0 = \sum_{i=1}^{n} d_i \left[\sum_{j=1}^{m} c_{ij}\mathbf{w}_j\right] = \sum_{j=1}^{m}\left[\sum_{i=1}^{n} d_i c_{ij}\right]\mathbf{w}_j.$$

Because the $\mathbf{w}_j$ are linearly independent, the only way to satisfy this equation is for

$$\sum_{i=1}^{n} d_i c_{ij} = 0, \quad j = 1, \ldots, m.$$

This is set of m linear, homogeneous equations in the n unknowns d_i, where the number of unknowns is greater than the number of equations. Hence, there must be an infinite number of nontrivial solutions. But this implies that the $\mathbf{v}_i$ are linearly dependent, a contradiction. Hence, there cannot be a smaller number of vectors that spans the space.∎

Because $\mathcal{G}^3$ and $\mathcal{R}^3$ are 3-dimensional vector spaces, any three linearly independent vectors constitute a basis. We cannot make the same claim about infinite-dimensional vector spaces; that is, we cannot say that any infinite set of linearly independent vectors constitutes a basis. For example, the space $P(x)$ of all polynomials in x is infinite-dimensional, and a basis is $\{1, x, x^2, x^3, \ldots\}$. The infinite set of even polynomials $\{1, x^2, x^4, \ldots\}$ is linearly independent, but it does not form a basis for the space. It would be a basis for the subspace of all even polynomials.

In the proof of Theorem 1.8, we verified the following corollary.

Corollary 1.8.1 When $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V, the component representation 1.5 for a vector is unique.

It is worth further emphasis to mention that whenever we describe a vector $\mathbf{v}$ by giving its components, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ say, then these components are with respect to some specified basis. If no mention of basis has been made, we assume that the components are with respect to the natural basis of the space, assuming that the natural basis for the space is clear. For instance, suppose that natural components of three vectors in $\mathcal{R}^3$ are $\mathbf{b}_1 = (1, 2, 3)$, $\mathbf{b}_2 = (-2, 5, 1)$, and $\mathbf{b}_3 = (4, 1, -2)$. Because these vectors are linearly independent, they can be used as a basis for the space. Relative to the basis $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$, the components of $\mathbf{b}_2$ are $(-2, 5, 1)$, but with respect to the basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, the components of $\mathbf{b}_2$ are $(0, 1, 0)$. Similarly, components of $\mathbf{b}_1$ and $\mathbf{b}_3$ with the respect to the basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are $(1, 0, 0)$ and $(0, 0, 1)$, respectively.

In equation 1.5, we have assumed that the space is n-dimensional. When it is infinite-dimensional, the situation may be exactly the same or quite different. For instance, space $P(x)$ of all polynomials is infinite-dimensional with basis $\{1, x, x^2, x^3, \ldots\}$. Every vector $p(x)$ in the space is a polynomial of some degree n,

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

The vector therefore has an infinite number of components $(a_0, a_1, \ldots, a_n, 0, 0, \ldots)$, but at most $n + 1$ are nonzero. Compare this with the infinite-dimensional vector space of all infinite sequences $\{c_1, c_2, \ldots\}$ of real numbers such that $\sum_{n=1}^{\infty} c_n^2$ converges. A basis for the space consists of the vectors e_i, $i \geq 1$ whose components are all zero except for the i^{th} one which is 1; that is,

$$\begin{array}{c}(n-1)^{\text{th}} \text{ component} \downarrow \quad \downarrow (n+1)^{\text{th}} \text{ component} \\ \mathbf{e}_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots).\end{array}$$

Vectors in this space can have an infinite number of nonzero components,

$$\mathbf{v} = c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 + c_3 \mathbf{e}_3 + \cdots . \tag{1.6}$$

This introduces a difficulty not found in finite-dimensional spaces, or in $P(x)$. Because expression 1.6 contains an infinite series, there is a question of convergence. We will deal with this later when we consider infinite-dimensional spaces in detail.

Example 1.15 Show that the vectors $(1, 2, -3)$, $(0, 4, -3)$, and $(3, 1, 5)$ constitute a basis for $\mathcal{R}^3$, and find the components of $\mathbf{v} = (2, -1, 6)$ with respect to this basis.

Solution The vectors form a basis if they are linearly independent. This is confirmed by the fact that

$$\det \begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 1 \\ -3 & -3 & 5 \end{pmatrix} = 41 \neq 0.$$

If (v_1, v_2, v_3) are the components of $(2, -1, 6)$ with respect to this basis, then

$$(2, -1, 6) = v_1(1, 2, -3) + v_2(0, 4, -3) + v_3(3, 1, 5).$$

When we equate components of these vectors, we obtain

$$\begin{aligned} v_1 + 3v_3 &= 2, \\ 2v_1 + 4v_2 + v_3 &= -1, \\ -3v_1 - 3v_2 + 5v_3 &= 6. \end{aligned}$$

The solution is $v_1 = -17/41$, $v_2 = -10/41$, and $v_3 = 33/41$. Thus,

$$(2, -1, 6) = -\frac{17}{41}(1, 2, -3) - \frac{10}{41}(0, 4, -3) + \frac{33}{41}(3, 1, 5).\bullet$$

Example 1.16 What is the most likely choice for the natural basis for the vector space $M_{2,2}(\mathcal{R})$ of all 2×2 real matrices?

Solution The matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are linearly independent (none is a linear combination of the others). Furthermore, they span the space since every 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a linear combination of them,

$$M = aM_1 + bM_2 + cM_3 + dM_4.$$

Since no smaller set of matrices can span the space, they constitute a basis, which we call the natural basis for $M_{2,2}(\mathcal{R})$. The space has dimension 4.$\bullet$

The extension of this example is to take as natural basis for vector space $M_{m,n}(\mathcal{R})$, the mn matrices $E_{i,j}$ of size $m \times n$ whose entries are all zero except for a one in the $(i,j)^{\text{th}}$ position. It would be necessary for us to have an understanding as to the order in which the basis matrices would be taken. We choose to take them in the order suggested by Example 1.16,

$$E_{1,1}, E_{1,2}, \ldots, E_{1,n}, E_{2,1}, E_{2,2}, \ldots, E_{2,n}, E_{3,1}, E_{3,2}, \ldots, E_{3,n}, \ldots, E_{m,n}. \tag{1.7}$$

Example 1.17 Vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ in $\mathcal{G}^3$ whose components satisfy $3v_x + 2v_y - v_z = 0$ form a subspace of dimension 2. Find a basis for the subspace. Show that the vector $\mathbf{v} = \langle -4, 1, -10 \rangle$ is in the subspace, and find its components with respect to your basis.

Solution Geometrically, the subspace consists of all vectors with tails at the origin and tips in the plane $3x + 2y - z = 0$. We need two linearly independent vectors in the plane. Since a normal to the plane is $\langle 3, 2, -1 \rangle$, vectors in the plane must be perpendicular to this vector. Two linearly independent ones are $\langle -2, 3, 0 \rangle$ and $\langle 0, 1, 2 \rangle$. Since the components of $\mathbf{v}$ satisfy the equation $3v_x + 2v_y - v_z = 0$, the vector is in the subspace. If v_1 and v_2 are the components of $\mathbf{v}$ with respect to the chosen basis of the subspace, then

$$\langle -4, 1, -10 \rangle = v_1 \langle -2, 3, 0 \rangle + v_2 \langle 0, 1, 2 \rangle.$$

When we equate components, we get

$$\begin{aligned} -2v_1 &= -4, \\ 3v_1 + v_2 &= 1, \\ 2v_2 &= -10. \end{aligned}$$

The solution is $v_1 = 2$ and $v_2 = -5$. Thus,

$$\langle -4, 1, -10 \rangle = 2\langle -2, 3, 0 \rangle - 5\langle 0, 1, 2 \rangle.\bullet$$

In this paragraph, we make an observation that will make your studies much more profound if you can constantly keep it in mind. Every vector space has a basis, and every vector $\mathbf{v}$ in the space has components with respect to that basis. If the space is n-dimensional, we write the components in the form $(v_1, v_2, \ldots, v_n)$. For example, if the space is 4-dimensional, then $(1, -2, 3, 5)$ would be the components of a vector in the space. Do **NOT** assume that this is a 4-tuple, and it is therefore a vector in $\mathcal{R}^4$ using the natural basis for $\mathcal{R}^4$. It could also be the vector $\mathbf{v} = (1,2,3,1) - 2(3,2,0,4) + 3(1,1,1,-1) + 5(3,5,7,0)$ in $\mathcal{R}^4$ using the basis $\{(1,2,3,1),(3,2,0,4),(1,1,1,-1),(3,5,7,0)\}$. Furthermore, it doesn't have to be a vector in $\mathcal{R}^4$; it could be a vector in **ANY** 4-dimensional space. For example, it could be the vector $p(x) = 1 - 2x + 3x^2 + 5x^3$ in $P_3(x)$, the space of polynomials of degree less than or equal to three, (using therefore the natural basis $\{1, x, x^2, x^3\}$). It could be the vector $p(x) = (1+x) - 2(3-x^2) + 3(x+x^3) + 5x^3$ in $P_3(x)$ using the basis $\{1+x, 3-x^2, x+x^3, x^3\}$. It could also be the matrix $\begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}$ in $M_{2,2}(\mathcal{R})$ (using the natural basis in Example 1.16). An n-tuple then, might represent the natural components of a vector in $\mathcal{R}^n$, but it could be the components of any vector in any n-dimensional vector space with respect to any basis for that space. Constantly keep this in mind. To begin with, Corollaries 1.3.1 and 1.4.1 and Theorems 1.5 and 1.6 are valid in more general spaces than in $\mathcal{G}^n$ and $\mathcal{R}^n$. Their more general statements are below.

Corollary 1.3.2 Two nonzero vectors in a vector space are linearly dependent if, and only if, each is a multiple of the other.

Corollary 1.4.2 Three nonzero vectors in a vector space are linearly dependent if, and only if, all three vectors are multiples of each other, or failing this, one of the vectors is a linear combination of the other two.

Theorem 1.9 A set of m vectors in an n-dimensional vector space, where $m > n$, is always linearly dependent. (In short, if you have more vectors than components, then the vectors are linearly dependent.)

Theorem 1.10 A set of n nonzero vectors in an n-dimensional vector space is linearly dependent if, and only if, the $n \times n$ determinant whose columns (or rows) are the components of the vectors has value 0.

Example 1.18 Redo Example 1.12 in Section 1.3 using Theorem 1.10.

Solution Components of the vectors with respect to the natural basis of $P_2(x)$ are $(1,1,0)$, $(2,3,-2)$, and $(0,5,-4)$. Since the determinant

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 5 \\ 0 & -2 & -4 \end{vmatrix} \neq 0,$$

the vectors are linearly independent.•

The Dimensions of Complex Vector Spaces

Many vector spaces that involve complex numbers can be regarded in conceptually different ways, and although eventual results are the same, calculations leading to these results may differ. We begin with the vector space of complex numbers itself. If we denote it by $\mathcal{C} = \mathcal{C}^1$, it is complex vector space with complex vectors and complex scalars. It has dimension 1, since a basis for the space is the complex number $z = 1$. Every vector (complex number) z can be expressed in the form $z\,1$; that is, as z times the basis vector 1. But we could also regard the space of complex numbers as a real vector space. Vectors are complex numbers and scalars are reals. In this case, a basis for the space is $\{1, i\}$. Every vector $a + bi$ in the space can be expressed in the form $a\,1 + bi$, where a and b are real. From this point of view, the dimension of the space is 2, and we should not denote it by $\mathcal{C}$.

In a similar way, we can regard the vector space of all pairs of complex numbers as a real vector space or a complex space. As a complex space, we denote it by $\mathcal{C}^2$, where scalars are complex numbers. As such, a basis for the space is the pair of vectors $(1,0)$ and $(0,1)$, and the space has dimension 2. Every complex pair (z_1, z_2) can be expressed in the form $z_1(1,0) + z_2(0,1)$. On the other hand, we could consider the space of complex pairs as a real vector space so that scalars are real. In this case, a basis is $\{(1,0),(i,0),(0,1),(0,i)\}$. Every complex pair $(z_1,z_2)=(a+bi,c+di)$ can be expressed in the form $a(1,0)+b(i,0)+c(0,1)+d(0,i)$. From this point of view, the space has dimension 4.

These ideas can carry over to other vector spaces involving complex numbers. For instance, consider the space $M_{2,2}(\mathcal{C})$, the space of complex 2×2 matrices. As a complex space it has dimension 4, being spanned by the matrices

$$\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Considered as a real vector space of 2×2 complex matrices with (real scalars), the space has dimension 8 with basis

$$\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}i&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&i\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\i&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&i\end{pmatrix}.$$

These ideas do not permeate every complex vector space. For instance, it would be pointless to attempt to regard the complex space $P_2(z)$ as a vector space over the reals.

EXERCISES 1.4

In Exercises 1–13 determine whether the set of vectors constitutes a basis for a vector space. In Exercises 1–10 take the dimension of the space to be the number of components of the vectors.

1. $\{(1,2),(3,5)\}$

2. $\{(2,1),(-4,2)\}$

3. $\{(1,3),(2,-3),(4,10)\}$

4. $\{(1,-2,4),(2,-5,0)\}$

5. $\{(2,-4,1),(1,-3,5),(5,-11,7)\}$

6. $\{(3,2,-1),(3,5,8),(-2,4,1)\}$

7. $\{(3,2,-1),(3,5,8),(-2,4,1),(1,1,1)\}$

8. $\{(1,-1,1,-1),(1,3,-2,5),(4,0,2,5)\}$

9. $\{(2,0,3,4),(1,-3,5,1),(1,0,0,3),(4,-3,8,8)\}$

10. $\{(2,0,3,4),(1,1,1,2),(2,4,1,3),(4,3,2,1)\}$

11. $\{5,2-3x,3+x-4x^2\}$ for $P_2(x)$

12. $\{x^{2n}\}$, where $n\geq 0$ is an integer, for $P(x)$

13. The matrices $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix}\right\}$ for $M_{2,2}(\mathcal{R})$.

14. Find the components of the polynomial $3+2x-5x^2$ with respect to the basis of polynomials for $P_2(x)$ in Exercise 11.

15. Find the components of the vector $(1,2,1,-1)$ with respect to the basis of vectors for $\mathcal{R}^4$ in Exercise 10.

16. Find the components of the polynomial $i+(1+i)z-3z^2$ with respect to the basis of polynomials $\{2-3i,iz,iz^2\}$ of $P_2(z)$.

17. (a) Show that the vectors $\mathbf{b}_1=\langle 1,2,0\rangle$ and $\mathbf{b}_2=\langle -1,4,4\rangle$ are a basis for the subspace W of vectors $\mathbf{v}=\langle v_x,v_y,v_z\rangle$ in $\mathcal{G}^3$ that satisfy $4v_x-2v_y+3v_z=0$.
(b) Verify that the vector $\mathbf{v}=\langle 3,9,2\rangle$ is in W, and find its components with respect to the basis in part (a).

18. If a set of vectors spans a space, must they be linearly independent?

19. Show that if $\{\mathbf{v}_1,\mathbf{v}_2\}$ is a basis for a vector space, so also is $\{\mathbf{v}_1+\mathbf{v}_2,\mathbf{v}_1-\mathbf{v}_2\}$.

20. If n vectors span a vector space, what can be said about the dimension of the space?

21. We have implicitly assumed that every basis of a finite-dimensional vector space contains the same number of vectors (the dimension of the space). Prove this.

22. Find a basis for the subspace of vectors (polynomials) $p(x)$ in $P_4(x)$ that satisfy

$$\int_{-1}^{1} p(x)\,dx = 0, \qquad p(0) = 0.$$

23. Is it possible to find four numbers a_1, a_2, a_3, and a_4 so that the following four polynomials are linearly independent?

$$\begin{aligned} p_1(x) &= (x - a_2)(x - a_3), \\ p_2(x) &= (x - a_3)(x - a_4), \\ p_3(x) &= (x - a_4)(x - a_1), \\ p_4(x) &= (x - a_1)(x - a_2). \end{aligned}$$

24. In the space $C^0(-\infty, \infty)$ of continuous functions, what is the dimension of the subspace spanned by the functions

$$\sin^2 x, \quad \cos^2 x, \quad \sin 2x, \quad \cos 2x?$$

25. Is the function $\sin 2x$ in the subspace of $C^0(-\infty, \infty)$ spanned by $\sin x$ and $\cos x$?

26. Find a basis for the vector space of real arithmetic sequences. What is the dimension of the space?

27. What is the dimension of the space $M_{m,n}(\mathcal{C})$ of $m \times n$ matrices with complex entries?

28. Prove that when W_1 and W_2 are subspace of a vector space V, then

$$\text{dimension } (W_1 \cap W_2) + \text{dimension } (W_1 + W_2) = \text{dimension } W_1 + \text{dimension } W_2.$$

Answers

1. Yes **2.** No **3.** No **4.** No **5.** No **6.** Yes **7.** No **8.** No **9.** Yes **10.** Yes **11.** Yes **12.** No **13.** No **14.** $(5/4, -1/4, -1/20)$ **15.** $(4, -15, 22/5, -1/5)$ **16.** $\big(3i, 1 - i, (-3 + 2i)/13\big)$ **17.**(b) $(7/2, 1/2)$ **18.** No **20.** Less than or equal to n **22.** $\{x, x^3\}$ **23.** No **24.** 3 **25.** No **26.** $(1, 1, 1, 1, \ldots)$, $(0, 1, 2, 3, \ldots)$, 2 **27.** mn

§1.5 Column Space, Row Space, Null Space, and Rank of a Matrix

Much of our work involves properties of matrices, in particular, whether the rows and columns of a matrix, considered as vectors, are linearly independent or linearly dependent. Consider therefore an $m \times n$ matrix $A = (a_{ij})_{m\times n}$. The columns of the matrix can be considered as vectors in $\mathcal{R}^m$ and the rows as vectors in $\mathcal{R}^n$, but they could also be vectors in any m- and n-dimensional spaces. When $m = n$, we have a test in Theorem 1.10 to determine whether the columns and rows constitute linearly independent sets of vectors. What is not clear, perhaps, is how many rows and how many columns are linearly independent when the complete set of them is linearly dependent. When $n > m$, the columns are linearly dependent, but it is not clear how many rows are linearly independent. Similarly, when $m > n$, the rows are linearly dependent, but how many columns are linearly independent is not known.

Definition 1.5 The **column space** of a matrix $A_{m\times n}$ is the space spanned by its columns, and the **row space** is the space spanned by its rows.

The column space of an $m \times n$ matrix is a subspace of some m-dimensional space; the row space is a subspace of some n-dimensional space. For example, if A is the matrix

$$A = \begin{pmatrix} -3 & -6 & 3 & 5 & 14 \\ -2 & -4 & 1 & 2 & 1 \\ 2 & 4 & 0 & -1 & 6 \end{pmatrix},$$

the column space is spanned by the vectors

$$\begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -6 \\ -4 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 14 \\ 1 \\ 6 \end{pmatrix},$$

and the row space is spanned by the vectors

$$(-3, -6, 3, 5, 14), \quad (-2, -4, 1, 2, 1) \quad \text{and} \quad (2, 4, 0, -1, 6).$$

Since there are five column vectors each with three components, we can say that the maximum dimension of the column space is three. This will be the case if three of the column vectors are linearly independent, but there could be fewer. Since there are three row vectors each with five components, we can say that the maximum dimension of the row space is three. It will be three if all three vectors are linearly independent, but it could be less if they are dependent. We would like to know the dimensions of these spaces, and be able to find bases for them. We first prove that their dimensions are the same, a result that is at first quite startling.

Theorem 1.13 The column and row spaces of a matrix have the same dimension.

Proof We begin with the row space. If $A = (a_{ij})$ is an $m \times n$ matrix, denote its i^{th} row by $\mathbf{r}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for the row space A. The value of k is as yet unknown, but it represents the dimension of the row space. Let the components of the i^{th} basis vector be denoted by $\mathbf{v}_i = (v_{i1}, v_{i2}, \ldots, v_{in})$. Each row vector can be expressed in terms of the $\mathbf{v}_i$,

$$\mathbf{r}_i = c_{i1}\mathbf{v}_1 + c_{i2}\mathbf{v}_2 + \cdots + c_{ik}\mathbf{v}_k, \quad i = 1, \ldots, m,$$

or,

$$\mathbf{r}_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix} = c_{i1}\begin{pmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{pmatrix} + c_{i2}\begin{pmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{pmatrix} + \cdots + c_{ik}\begin{pmatrix} v_{k1} \\ v_{k2} \\ \vdots \\ v_{kn} \end{pmatrix}, \quad i = 1, \ldots, m.$$

Suppose we take the first of these ($i = 1$) and equate the j^{th} components, do the same for the second ($i = 2$), the third, to the m^{th}. The result is the m equations

$$\begin{aligned}
a_{1j} &= c_{11}v_{1j} + c_{12}v_{2j} + \cdots + c_{ik}v_{kj},\\
a_{2j} &= c_{21}v_{1j} + c_{22}v_{2j} + \cdots + c_{2k}v_{kj},\\
\vdots &\quad\ \ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots\\
a_{mj} &= c_{mj}v_{1j} + c_{m2}v_{2j} + \cdots c_{mk}v_{kj}.
\end{aligned}$$

We can write these in the form

$$\begin{pmatrix} a_{1i} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = v_{1j} \begin{pmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{pmatrix} + v_{2j} \begin{pmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{m2} \end{pmatrix} + \cdots + v_{kj} \begin{pmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{mk} \end{pmatrix}.$$

Now, the left side of this equation is the j^{th} column of A. The right side is a linear combination of k vectors. Since this is true for each of the columns of A, we can conclude that the dimension of the column space of A is less than or equal to k; that is,

$$\text{dimension of column space of } A \leq \text{dimension of the row space of } A.$$

But by a similar argument, we can show that

$$\text{dimension of row space of } A \leq \text{dimension of the column space of } A.$$

We must conclude that the dimension of the row space is equal to the dimension of the column space.∎

Definition 1.6 The dimension of the row and column spaces of a matrix A is called the **rank** of the matrix, denoted by $\text{rank}(A)$.

The next theorem tells us how to find the rank of a matrix, and how to find a basis for the row space of the matrix.

Theorem 1.14 When A_{rref} is the reduced row echelon form of a matrix A, the rank of A is the number of nonzero rows in A_{rref}, and the nonzero rows form a basis for the row space of A.

Proof Since the rows of A_{rref} are linear combinations of the rows of A, they must span the row space of A. All that we need show is that they are linearly independent vectors. Denote the nonzero rows, from top to bottom, by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$, where k is therefore the rank of A. Consider finding scalars $c_1, c_2, \ldots, c_k$ so that

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + \cdots + c_k\mathbf{r}_k = \mathbf{0}.$$

The first component of $\mathbf{r}_1$ is 1, and the first component of the remaining vectors is zero. If we equate the first component of left and right sides of this equation, we get $c_1 = 0$. We now have

$$c_2\mathbf{r}_2 + c_3\mathbf{r}_3 + \cdots + c_k\mathbf{r}_k = \mathbf{0}.$$

The first nonzero component of $\mathbf{r}_2$ is 1, and the corresponding component of the remaining vectors is zero. If we equate this component of left and right sides of this equation, we get $c_2 = 0$. Continuing with this procedure shows that all the c_i are equal to zero. Hence, the rows of A_{rref} are linearly independent, and constitute a basis for the row space of A.∎

Corollary 1.14.1 The rows in the reduced row echelon form of the transpose A^T constitute a basis for the column space of A.

Here are two examples to illustrate.

Example 1.19 Find the rank, and bases for the row and column spaces of the matrix

$$A = \begin{pmatrix} -3 & -6 & 3 & 5 & 14 \\ -2 & -4 & 1 & 2 & 1 \\ 2 & 4 & 0 & -1 & 6 \end{pmatrix}.$$

Solution The reduced row echelon form for this matrix is

$$A_{\text{rref}} = \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

The matrix has rank 3, and the three vectors $(1,2,0,0,5)$, $(0,0,1,0,3)$, and $(0,0,0,1,4)$ constitute a basis for the row space. The original three rows of A also constitute a basis for the row space, but the reduced row echelon form of A has produced a simpler basis. Because the column space has dimension three, and we can think of vectors in the column space as vectors in $\mathcal{R}^3$, it follows that the column space is $\mathcal{R}^3$. We can therefore choose the natural basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ as a basis for the column space. Had we taken the trouble of finding the reduced row echelon form for A^T, we would have obtained the same basis.•

Example 1.20 Find the rank, and bases for the row and column spaces of the matrix

$$A = \begin{pmatrix} 3 & 4 & -1 & 4 \\ 2 & 1 & 5 & -6 \\ 4 & 7 & 2 & 1 \\ 9 & 10 & 6 & -1 \end{pmatrix}.$$

Solution The reduced row echelon form for this matrix is

$$A_{\text{rref}} = \begin{pmatrix} 1 & 0 & 0 & 5/7 \\ 0 & 1 & 0 & 13/77 \\ 0 & 0 & 1 & -117/77 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The dimension of the row space is three, and this must also be the dimension of the column space, and the rank of the matrix. The three vectors $(1,0,0,5/7)$, $(0,1,0,13/77)$, and $(0,0,1,-177/77)$ constitute a basis for the row space. So also do the vectors $(7,0,0,5)$, $(0,77,0,13)$, and $(0,0,77,-177)$. The reduced row echelon form for A^T is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A basis for the column space is $\{(1,0,0,1),(0,1,0,1),(0,0,1,1)\}$.•

Any set of vectors in a vector space spans a subspace. We can use the above ideas to determine whether the vectors are linearly independent, and at the same time find a basis for the subspace.

Example 1.21 Find a basis for the subspace of a 4-dimensional space spanned by the vectors

$$(-2,3,4,1), \quad (3,0,2,5), \quad (1,3,6,6), \quad (-5,3,2,-4).$$

Solution We form a matrix with these vectors as rows,

$$A = \begin{pmatrix} -2 & 3 & 4 & 1 \\ 3 & 0 & 2 & 5 \\ 1 & 3 & 6 & 6 \\ -5 & 3 & 2 & -4 \end{pmatrix}.$$

The reduced echelon form for this matrix is

$$A_{\text{rref}} = \begin{pmatrix} 1 & 0 & 2/3 & 5/3 \\ 0 & 1 & 16/9 & 13/9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This shows that only two of the vectors were linear independent. The subspace spanned by the vectors has dimension two, and a basis is $(3, 0, 2, 5)$ and $(0, 9, 16, 13)$.●

As a final consideration in this section, we relate some of the above ideas to solutions of systems of linear equations, and at the same time identify a third space associated with a matrix. A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ can be represented in matrix form as

$$A\mathbf{x} = \mathbf{b}, \tag{1.8a}$$

where

$$A = (a_{ij})_{m,n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}. \tag{1.8b}$$

The system is said to be **homogeneous** if $\mathbf{b} = \mathbf{0}$, **nonhomogeneous** otherwise. The homogeneous system

$$A\mathbf{x} = \mathbf{0} \tag{1.9}$$

always has the trivial solution $\mathbf{x} = \mathbf{0}$, but it may have others. The set of solutions is called the **null space** of matrix A, and as the name suggests, the set of solutions constitutes a vector space. The dimension of the null space of a matrix is often called the **nullity** of the matrix. The essential question of whether the null space contains more than the zero vector is answered in the following theorem.

Theorem 1.15 **(Rank-Nullity Theorem)** If an $m \times n$ matrix A has rank r, then its null space has dimension $n - r$. If we denote the nullity of A by N, then

$$N + r = n. \tag{1.10a}$$

In words,

$$\begin{aligned} &\text{dimension(null space of } A\text{)+dimension(column space of } A\text{)} \\ &\qquad\qquad = \text{number of columns of } A. \end{aligned} \tag{1.10b}$$

Proof Because the rank of A is r, its reduced row echelon form A_{rref} has r leading ones. We can find all solutions of $A\mathbf{x} = \mathbf{0}$ by solving the system $A_{\text{rref}}\mathbf{x} = \mathbf{0}$ for the r variables corresponding to the leading 1's in terms of the remaining $n - r$ so called free variables. When $n > m$, the maximum value of r is m, and $n - r$ must be positive. In this case, the $n - r$ family of solutions constitutes a vector space of dimension $n - r$. When $n \leq m$, the maximum value of r is n. When $n = r$, the only solution is the trivial one, and when $r < n$, there is once again an $(n - r)$-dimensional space of solutions.∎

Example 1.22 Find a basis for the null space of the matrix in Example 1.20.

Solution The null space consists of vectors satisfying

$$A\mathbf{x} = \mathbf{0} \qquad \Longleftrightarrow \qquad \begin{pmatrix} 3 & 4 & -1 & 4 \\ 2 & 1 & 5 & -6 \\ 4 & 7 & 2 & 1 \\ 9 & 10 & 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The reduced row echelon form for the augmented matrix of the system is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 5/7 & 0 \\ 0 & 1 & 0 & 13/77 & 0 \\ 0 & 0 & 1 & -117/77 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Vectors in the null space therefore satisfy

$$x_1 = -\frac{5x_4}{7}, \quad x_2 = -\frac{13x_4}{77}, \quad x_3 = \frac{117x_4}{77};$$

that is, they take the form

$$\mathbf{x} = \begin{pmatrix} -5x_4/7 \\ -13x_4/77 \\ 117x_4/77 \\ x_4 \end{pmatrix} = -\frac{x_4}{77}\begin{pmatrix} 5 \\ 13 \\ -117 \\ -77 \end{pmatrix}.$$

Hence, the null space has dimension 1 and a basis vector is $(5, 13, -117, -77)$,•

Now consider when system 1.8 is nonhomogeneous,

$$A\mathbf{x} = \mathbf{b}, \tag{1.11}$$

where $\mathbf{b} \neq \mathbf{0}$. We can rewrite the system in the form,

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}. \tag{1.12}$$

Since the matrices on the left are the column vectors of A, we can say that there is a solution of the system if, and only if, vector $\mathbf{b}$ is in the column space of A. When $\mathbf{b}$ is not in the column space of A, there is no solution of the equations. When $\mathbf{b}$ is in the column of space of A, there may be one solution or an infinite number of solutions of the equations. They are described in the following theorem.

Theorem 1.16 If the nonhomogeneous system of equations $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}_p$, then every solution of the system can be expressed in the form $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof Let $\mathbf{x}$ be any solution of $A\mathbf{x} = \mathbf{b}$. Then

$$A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0};$$

that is, $\mathbf{x} - \mathbf{x}_p$ satisfies the homogeneous system $A\mathbf{x} = \mathbf{0}$. Hence $\mathbf{x} - \mathbf{x}_p$ must be equal to $\mathbf{x}_h$, for some $\mathbf{x}_h$; that is, $\mathbf{x} - \mathbf{x}_p = \mathbf{x}_h$, or, $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$. In the event that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, the nonhomogeneous system will have an infinity of solutions, and when the homogeneous system has only the trivial solution, the nonhomogeneous has a unique solution.■

When homogeneous system 1.9 has an $n - r$ parameter of solutions, so also does nonhomogeneous system 1.11, but the solutions do not constitute a vector a space.

EXERCISES 1.5

In Exercises 1–7 find: (a) the rank of the matrix, (b) a basis for its row space, (c) a basis for its column space, and (d) a basis for its null space.

1. $\begin{pmatrix} 3 & 3 & 2 & -1 \\ 3 & 0 & 2 & 1 \\ 1 & 3 & 5 & 6 \\ 7 & 6 & 9 & 6 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 3 & 1 & 7 \\ 3 & 0 & 3 & 6 \end{pmatrix}$

3. $\begin{pmatrix} 3 & 2 & 1 \\ -2 & 4 & 1 \\ 5 & -2 & 0 \\ 9 & 6 & 3 \end{pmatrix}$

4. $\begin{pmatrix} 2 & 1 & -2 & 3 \\ 5 & 2 & 3 & -6 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 2 & 4 & 0 & -3 \\ 0 & 1 & -5 & -1 & 2 \\ 0 & 1 & 3 & -1 & 0 \\ 2 & 0 & -1 & -1 & 0 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & 3 & 1 & 0 \\ 1 & 9 & 10 & 11 \end{pmatrix}$

7. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & 3 & 1 & 0 \\ 1 & 9 & 10 & 12 \end{pmatrix}$

In Exercises 8–11 the set of vectors spans a subspace in a 4-dimensional space. Find a basis for the subspace.

8. $\{(2,3,-1,1),(3,-4,-2,-1),(1,1,2,2),(3,4,5,6)\}$

9. $\{(2,3,-1,1),(3,-4,-2,-1),(1,1,2,2),(6,0,-1,2)\}$

10. $\{(2,3,-1,1),(3,-4,-2,-1),(-1,7,1,2),(4,6,-2,2)\}$

11. $\{(2,3,5,-1),(3,-4,-1,7),(-1,-2,-3,1)\}$

12. If we are not particularly interested in a simple basis for the column space of a real matrix A, there is an alternative to finding the reduced row echelon form for A^T. Take the columns of A corresponding to the columns in A_{rref} that have leading ones. What basis does this lead to for the matrix in Exercise 4. Confirm that these vectors do indeed form a basis for the column space of A.

13. Use the matrix

$$\begin{pmatrix} 1-i & i \\ 2 & -1+i \end{pmatrix}$$

to illustrate that the technique of Exercise 12 for finding a basis for the column space of a matrix does not work for complex matrices.

Answers

1.(a) 3 (b) $(13,0,0,-11)$, $(0,3,0,-2)$, $(0,0,13,23)$ (c) $(1,0,0,1)$, $(0,1,0,1)$, $(0,0,1,1)$ (d) $(33,26,-69,30)$
2.(a) 2 (b) $(1,0,1,2)$, $(0,3,-2,1)$ (c) $(1,0)$, $(0,1)$ (d) $(-3,2,3,0)$, $(6,1,0,-3)$
3.(a) 2 (b) $(8,0,1)$, $(0,16,5)$ (c) $(1,0,1,3)$, $(0,1,-1,0)$ (d) $(2,5,-16)$
4.(a) 3 (b) $(5,0,0,-11)$, $(0,5,0,23)$, $(0,0,5,-7)$ (c) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (d) $(11,-23,7,5)$
5.(a) 4 (b) $(5,0,0,0,-4)$, $(0,5,0,0,-3)$, $(0,0,4,0,-1)$, $(0,0,0,20,-27)$ (c) $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ (d) $(7,12,14,0,29)$, $(12,4,-5,29,0)$
6.(a) 3 (b) $(1,0,1,0)$,$(0,1,1,0)$, $(0,0,0,1)$ (c) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (d) $(1,1,-1,0)$
7.(a) 2 (b) $(7,0,7,12)$, $(0,7,7,8)$ (c) $(1,0,3)$, $(0,1,1)$ (d) $(1,1,-1,0)$, $(12,8,0,-7)$
8. $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ **9.** $(43,0,0,19)$, $(0,43,0,11)$, $(0,0,43,28)$
10. $(17,0,-10,1)$, $(0,17,1,5)$ **11.** $(1,0,1,1)$, $(0,1,1,-1)$ **12.** $(2,5,1)$, $(1,2,1)$, $(-2,3,1)$

§1.6 Changing Bases

The choice of basis for a vector space can make a huge difference on how easy it is to solve a problem. At the moment, we might feel that the natural basis is optimum for $\mathcal{G}^3$ and $\mathcal{R}^3$, and for some problems we would be correct. On the other hand, there are problems in these spaces where the natural basis is not the best choice. Much of this course is about finding the best choice for the basis of a vector space, and as we suggested, it may vary from problem to problem. Once we have located what we feel is the best basis for a particular problem, it will invariably be necessary to transform components of vectors with respect to some basis to components with respect to the optimum basis, often from the natural basis to the optimum basis.

We begin with a numerical example in which we change from natural components of a vector in $\mathcal{R}^3$ to components with respect to a different basis. Suppose that a vector $\mathbf{v}$ has natural components $(4,-2,3)$; in other words, $\mathbf{v} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. The vectors $\mathbf{b}_1 = (1,2,3)$, $\mathbf{b}_2 = (-2,1,4)$, and $\mathbf{b}_3 = (-3,3,2)$ are linearly independent and therefore constitute another basis for $\mathcal{R}^3$. If (c_1,c_2,c_3) are the components of $\mathbf{v}$ with respect to this new basis, then

$$\begin{aligned}(4,-2,3) &= c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = c_1(1,2,3) + c_2(-2,1,4) + c_3(-3,3,2)\\ &= (c_1 - 2c_2 - 3c_3, 2c_1 + c_2 + 3c_3, 3c_1 + 4c_2 + 2c_3).\end{aligned}$$

When we equate components, we obtain the equations

$$\begin{aligned}c_1 - 2c_2 - 3c_3 &= 4,\\ 2c_1 + c_2 + 3c_3 &= -2,\\ 3c_1 + 4c_2 + 2c_3 &= 3.\end{aligned}$$

The solution of these is $c_1 = 33/35$, $c_2 = 29/35$, and $c_3 = -11/7$. The components of vector $\mathbf{v} = (4,-2,3)$ with respect to the basis $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are $33/35$, $29/35$, and $-11/7$; that is,

$$\mathbf{v} = \frac{33}{35}\mathbf{b}_1 + \frac{29}{35}\mathbf{b}_2 - \frac{11}{7}\mathbf{b}_3 = \frac{33}{35}(1,2,3) + \frac{29}{35}(-2,1,4) - \frac{11}{7}(-3,3,2).$$

This process could be repeated for any other vector with natural components (x_1,x_2,x_3); replace 4, -2, and 3 with x_1, x_2, and x_3,

$$\begin{aligned}c_1 - 2c_2 - 3c_3 &= x_1,\\ 2c_1 + c_2 + 3c_3 &= x_2,\\ 3c_1 + 4c_2 + 2c_3 &= x_3,\end{aligned}$$

and re-solve the system. If components of a large number of vectors were to be transformed, it would obviously be beneficial to invert the matrix of coefficients

$$A = \begin{pmatrix} 1 & -2 & -3\\ 2 & 1 & 3\\ 3 & 4 & 2\end{pmatrix},$$

and express the solution in the form

$$\begin{pmatrix} c_1\\ c_2\\ c_3\end{pmatrix} = A^{-1}\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}.$$

The inverse is $A^{-1} = \begin{pmatrix} 2 & -16/5 & 9/5\\ -1 & 7/5 & -3/5\\ -1 & 2 & -1\end{pmatrix}$, so that

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 & -16/5 & 9/5 \\ -1 & 7/5 & -3/5 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

What we should notice here is that the columns of A are the natural components of the basis vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ and even more importantly, the columns of A^{-1} are the components of the basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ with respect to the $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ basis; that is,

$$\hat{\mathbf{i}} = 2\mathbf{b}_1 - \mathbf{b}_2 - \mathbf{b}_3, \quad \hat{\mathbf{j}} = -\frac{16}{5}\mathbf{b}_1 + \frac{7}{5}\mathbf{b}_2 + 2\mathbf{b}_3, \quad \hat{\mathbf{k}} = \frac{9}{5}\mathbf{b}_1 - \frac{3}{5}\mathbf{b}_2 - \mathbf{b}_3.$$

This is not a coincidence of this example, nor does it depend on the fact that we began with natural components, nor does it depend on working in $\mathcal{R}^3$. This is confirmed in the following theorem.

Theorem 1.17 Suppose that $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$ are bases for a vector space V. The components of any vector $\mathbf{v}$ in V with respect to these bases are related by the equation

$$\mathbf{v_d} = T_{db}\mathbf{v_b}, \tag{1.13}$$

where columns of matrix T_{db} are the components of the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ with respect to the basis $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$. ($\mathbf{v_b}$ and $\mathbf{v_d}$ are the column matrix representations of the components of $\mathbf{v_b}$ and $\mathbf{v_d}$.)

Matrix T_{db} is called the **transition matrix** or the **change of basis matrix**. The above discussion also suggests the following corollary.

Corollary 1.17.1 When T_{db} is the change of basis matrix from one basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ to a second basis $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$, then its inverse is the change of basis matrix from $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$ to $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$; that is, $T_{bd} = T_{db}^{-1}$.

Example 1.23 Find the transition matrix from the natural basis in $\mathcal{R}^3$ to the basis consisting of the vectors $(1, 2, 3)$, $(-2, 0, 5)$, and $(0, 3, 1)$.

Solution There are two ways that we could proceed. One is to use Theorem 1.17 and find components of the natural basis with respect to the other basis. Alternatively, we can use the corollary to the theorem. The transition matrix from the second basis to the natural basis is

$$T = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 3 \\ 3 & 5 & 1 \end{pmatrix}.$$

The transition matrix from the natural basis to the other basis is the inverse of this matrix,

$$T^{-1} = \frac{1}{29}\begin{pmatrix} 15 & -2 & 6 \\ -7 & -1 & 3 \\ -10 & 11 & -4 \end{pmatrix}. \bullet$$

Example 1.24 Let $\mathbf{b}_1 = (1, 2)$ and $\mathbf{b}_2 = (-2, 3)$ be a basis for $\mathcal{R}^2$, and $\mathbf{d}_1 = (-2, 1)$ and $\mathbf{d}_2 = (1, 1)$ be a second basis. Find the transition matrix T_{db}. If $\mathbf{v_b} = (1, -5)$, what is $\mathbf{v_d}$?

Solution According to Theorem 1.17, the columns of T_{db} are the components of the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ with respect to the basis $\mathbf{d}_1$ and $\mathbf{d}_2$. If (b_{11}, b_{12}) are the components of $\mathbf{b}_1$ with respect to $\mathbf{d}_1$ and $\mathbf{d}_2$, then

$$(1, 2) = b_{11}(-2, 1) + b_{12}(1, 1) = (-2b_{11} + b_{12}, b_{11} + b_{12}).$$

When we equate components, we obtain

$$-2b_{11} + b_{12} = 1, \quad b_{11} + b_{12} = 2.$$

The solution is $b_{11} = 1/3$ and $b_{12} = 5/3$. A similar calculation shows that the components of $\mathbf{b}_2$ with respect to $\mathbf{d}_1$ and $\mathbf{d}_2$ are $b_{21} = 5/3$ and $b_{22} = 4/3$. The matrix T_{db} is

$$T_{db} = \begin{pmatrix} 1/3 & 5/3 \\ 5/3 & 4/3 \end{pmatrix}.$$

The components of $\mathbf{v_b} = (1, -5)$ with respect to the basis $\mathbf{d}_1$ and $\mathbf{d}_2$ are

$$(\mathbf{v_d}) = T_{db}\,(\mathbf{v_b}) = \begin{pmatrix} 1/3 & 5/3 \\ 5/3 & 4/3 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} -8 \\ -5 \end{pmatrix};$$

that is, $\mathbf{v_b} = -8(-2, 1) - 5(1, 1)$. We can confirm this by showing that natural components of $\mathbf{v_b}$ and $\mathbf{v_d}$ are the same,

$$\mathbf{v_b} = 1(1, 2) - 5(-2, 3) = (11, -13) \quad \text{and} \quad \mathbf{v_d} = -8(-2, 1) - 5(1, 1) = (11, -13).\bullet$$

In Example 1.23, we saw that it is straightforward to find the transition matrix from the natural basis to another basis, invert the matrix whose rows are components of the other basis. In Example 1.24, where neither basis was natural, we found components of the original basis vectors with respect to the ultimate basis. We could streamline the approach as follows. In finding components of $\mathbf{b}_1$ with respect to $\mathbf{d}_1$ and $\mathbf{d}_2$, we solved the system

$$-2b_{11} + b_{12} = 1, \quad b_{11} + b_{12} = 2.$$

In finding components of $\mathbf{b}_2$ with respect to $\mathbf{d}_1$ and $\mathbf{d}_2$, we solved the system

$$-2b_{21} + b_{22} = -2, \quad b_{21} + b_{22} = 3.$$

We can solve both systems simultaneously by reducing the first 2×2 matrix below to the identity,

$$\left(\begin{array}{cc|cc} -2 & 1 & 1 & -2 \\ 1 & 1 & 2 & 3 \end{array}\right).$$

The first two columns contain the components of the ultimate basis $\mathbf{d}_1$ and $\mathbf{d}_2$, and the last two columns contain components of the original basis $\mathbf{b}_1$ and $\mathbf{b_2}$. The result is

$$\left(\begin{array}{cc|cc} 1 & 0 & 1/3 & 5/3 \\ 0 & 1 & 5/3 & 4/3 \end{array}\right).$$

The last two columns contain components of $\mathbf{b_1}$ and $\mathbf{b_2}$ with respect to $\mathbf{d}_1$ and $\mathbf{d_2}$, the transition matrix. We illustrate this again in the next example.

Example 1.25 Find the transition matrix from the basis $(1, 2, 3)$, $(-3, 2, 5)$, and $(2, 0, 3)$ in $\mathcal{R}^3$ to the basis $(0, 2, 1)$, $(4, 1, -1)$, and $(1, 1, 0)$.

Solution We reduce the first 3×3 matrix in

$$\left(\begin{array}{ccc|ccc} 0 & 4 & 1 & 1 & -3 & 2 \\ 2 & 1 & 1 & 2 & 2 & 0 \\ 1 & -1 & 0 & 3 & 5 & 3 \end{array}\right)$$

to the identity. The result is

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 10 & 11 \\ 0 & 1 & 0 & 5 & 5 & 8 \\ 0 & 0 & 1 & -19 & -23 & -30 \end{array}\right),$$

and therefore the transition matrix is

$$\begin{pmatrix} 8 & 10 & 11 \\ 5 & 5 & 8 \\ -19 & -23 & -30 \end{pmatrix}. \bullet$$

The next two examples are more abstract.

Example 1.26 The natural basis for the space $P_2(x)$ of real polynomials of degree less than or equal to two is $\{1, x, x^2\}$. Find the transition matrix to the basis $\{3x, 1-2x, 2+10x+5x^2\}$.

Solution We should perhaps confirm that $\{3x, 1-2x, 2+10x+5x^2\}$ is indeed a basis for $P_2(x)$. Natural components of these vectors are $(0,3,0)$, $(1,-2,0)$, and $(2,10,5)$. Since

$$\det \begin{vmatrix} 0 & 1 & 2 \\ 3 & -2 & 10 \\ 0 & 0 & 5 \end{vmatrix} = -15 \neq 0,$$

the vectors are linearly independent and therefore form a basis. We can find the transition matrix in a number of ways.

Method 1 Columns of the transition matrix are the components of 1, x, and x^2 with respect to the other basis. We could get them as we did in Example 1.24, which for 1 would be to set

$$1 = c_1(3x) + c_2(1-2x) + c_3(2+10x+5x^2),$$

equate coefficients of 1, x, and x^2, and solve for c_1, c_2, and c_3. Then do this for x and x^2. This is perhaps the least efficient method.

Method 2 We can find the inverse of the transition matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 3 & -2 & 10 \\ 0 & 0 & 5 \end{pmatrix}$$

from the basis $\{3x, 1-2x, 2+10x+5x^2\}$ to the natural basis.

Method 3 Following the method of Example 1.25, we can reduce the first 3×3 matrix in

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & -2 & 10 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right)$$

to the identity. The calculations are those of Method 2.

Method 4 We can find the components in Method 1 by writing

$$\begin{aligned} 1 &= (1-2x) + 2x = (1-2x) + \frac{2}{3}(3x) = \frac{2}{3}(3x) + 1(1-2x), \\ x &= \frac{1}{3}(3x), \\ x^2 &= \frac{1}{5}(2+10x+5x^2) - 2x - \frac{2}{5} = \frac{1}{5}(2+10x+5x^2) - \frac{2}{5}(1-2x) - \frac{14}{5}x \\ &= \frac{1}{5}(2+10x+5x^2) - \frac{2}{5}(1-2x) - \frac{14}{15}(3x) \\ &= -\frac{14}{15}(3x) - \frac{2}{5}(1-2x) + \frac{1}{5}(2+10x+5x^2). \end{aligned}$$

These display the components of the natural basis with respect to the other basis., All methods lead to the transition matrix

$$T = \begin{pmatrix} 2/3 & 1/3 & -14/15 \\ 1 & 0 & -2/5 \\ 0 & 0 & 1/5 \end{pmatrix}. \bullet$$

Example 1.27 The vector space of solutions of the linear, homogeneous, second order differential equation

$$\frac{d^2y}{dx^2} - k^2y = 0,$$

where $k > 0$ is a constant, is 2-dimensional with basis $\{e^{kx}, e^{-kx}\}$. A second basis is the functions $\{\cosh kx, \sinh kx\}$. Find the transition matrix from the exponential basis to the hyperbolic basis.

Solution The columns of the matrix are the components of e^{kx} and e^{-kx} with respect to $\cosh kx$ and $\sinh kx$. Definitions of $\cosh kx$ and $\sinh kx$ are

$$\cosh kx = \frac{e^{kx} + e^{-kx}}{2} \qquad \text{and} \qquad \sinh kx = \frac{e^{kx} - e^{-kx}}{2},$$

and these imply that

$$e^{kx} = \cosh kx + \sinh kx \qquad \text{and} \qquad e^{-kx} = \cosh kx - \sinh kx.$$

Hence, the change of basis matrix is

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We could test this with a simple example. Consider the solution $5e^{kx} - 3e^{-kx}$. Its components with respect to the hyperbolic basis are

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

In other words, the solution is

$$2\cosh kx + 7\sinh kx = 2\left(\frac{e^{kx} + e^{-kx}}{2}\right) + 8\left(\frac{e^{kx} - e^{-kx}}{2}\right) = 5e^{kx} - 3e^{-kx}. \bullet$$

In each of the above examples, the transition matrix had constant entries. The following example illustrates that this may not always be the case.

Example 1.28 We have agreed to draw all vectors in $\mathcal{G}^2$ with their tails at the origin. We could have drawn them all with tails at some other fixed point, but discussions would have been the same, and the reason for this is that basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the same at any point in the plane. As a result, the natural components of a vector at one point are the same as the natural components at any other point. Basis vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ for polar coordinates do not share this constancy. They change direction from point to point (see Figure 1.1). What this means is that the polar components of a directed line segment at one point are different than the polar components at another point. The transition matrix from natural components to polar components will therefore vary from point to point. Find the matrix.

Solution The columns in the transition matrix are the components of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ with respect to $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. They will vary from point to point. Either algebraically, or using Figure 1.2, we find that

$$\hat{\mathbf{i}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}, \qquad \hat{\mathbf{j}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}.$$

The transition matrix is therefore

$$T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};$$

it depends on polar coordinate θ of the point at which vectors are drawn, but not the distance r from the origin to the point.

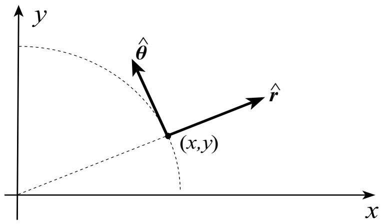

Figure 1.1

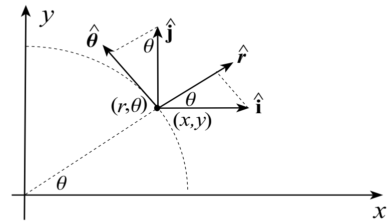

Figure 1.2

For example, consider finding the polar coordinates of the vector $\hat{\mathbf{i}}$ at the point with Cartesian coordinates $(2,0)$. Since $\theta = 0$ at this point, the polar coordinates of $\hat{\mathbf{i}}$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

that is, $\hat{\mathbf{i}} = \hat{\mathbf{r}}$. At the point $(4,4)$, the value of θ is $\pi/4$, and the components of $\hat{\mathbf{i}}$ at this point are

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix};$$

that is, $\hat{\mathbf{i}} = (\hat{\mathbf{r}} - \hat{\boldsymbol\theta})/\sqrt{2}$. Finally, at the point $(0,5)$ where $\theta = \pi/2$, the components of $\hat{\mathbf{i}}$ are

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix};$$

that is, $\hat{\mathbf{i}} = -\hat{\boldsymbol\theta}$. Draw $\hat{\mathbf{i}}$, $\hat{\mathbf{r}}$, and $\hat{\boldsymbol\theta}$ at the points $(2,0)$, $(4,4)$, and $(0,5)$ to confirm the validity of the results. It would also be a good idea to change components of $\hat{\mathbf{j}}$ at these same points, and to change components of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ at points in the other quadrants.

EXERCISES 1.6

In Exercises 1–8 find the transition matrix from the first basis to the second. Then find components of the given vector with respect to both bases and confirm that they are indeed correct. Assume that the components of the vector are natural components.

1. Natural basis to $\{(1,3),(3,5)\}$ in $\mathcal{R}^2$; $\mathbf{v} = (4,-7)$

2. $\{(1,1),(3,-2)\}$ to $\{(1,3),(3,5)\}$ in $\mathcal{R}^2$; $\mathbf{v} = (-3,8)$

3. $\{\langle 1,1,2\rangle, \langle 2,-1,0\rangle, \langle 0,3,-2\rangle\}$ to natural basis in $\mathcal{G}^3$; $\mathbf{v} = \langle -2,3,6\rangle$

4. Natural basis to $\{\langle 1,1,2\rangle, \langle 2,-1,0\rangle, \langle 0,3,-2\rangle\}$ in $\mathcal{G}^3$; $\mathbf{v} = \langle -3,2,5\rangle$

5. $\{(1,1,2),(2,-1,0),(0,3,-2)\}$ to $\{(3,1,-2),(1,1,0),(4,-3,2)\}$ in $\mathcal{R}^3$; $\mathbf{v} = (3,3,7)$

6. Natural basis to $\{3, 2-x, 4+x^2\}$ in $P_2(x)$; $p(x) = 3x^2 - 4x + 7$

7. Natural basis $\{1, z, z^2\}$ to $\{i, 3z, 2 - z + (1+i)z^2\}$ in $P_2(z)$; $p(z) = 2i + 4z + (2+3i)z^2$

8. Natural basis to basis $\{2, 3x, 4x^2, \ldots, (k+2)x^k, \ldots\}$ of $P(x)$; $p(x) = x^6 + 4x^3 - 2$

9. The first five Legendre polynomials are

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \frac{1}{2}(3x^2 - 1), \quad p_3(x) = \frac{1}{2}(5x^3 - 3x), \quad p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

The first four Chebyshev polynomials are

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1.$$

Both form a basis for $P_4(x)$.
(a) Find the transition matrix from the Legendre basis to the Chebyshev basis.
(b) Find the transition matrix from the Chebyshev basis to the Legendre basis.

10. Repeat Exercise 9 but replace the Chebyshev polynomials with the Hermite polynomials

$$h_0(x) = 1, \quad h_1(x) = 2x, \quad h_2(x) = 4x^2 - 2, \quad h_3(x) = 8x^3 - 12x, \quad h_4(x) = 16x^4 - 48x^2 + 12.$$

Answers

1. $\frac{1}{4}\begin{pmatrix} -5 & 3 \\ 3 & -1 \end{pmatrix}$; $(4, -7)$, $(-41/4, 19/4)$ **2.** $\frac{1}{4}\begin{pmatrix} -2 & -21 \\ 2 & 11 \end{pmatrix}$; $(18/5, -11/5)$, $(39/4, -17/4)$

3. $\begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 3 \\ 2 & 0 & -2 \end{pmatrix}$; $\langle 10/9, -20/9, -5/9 \rangle$, $\langle -2, 3, 6 \rangle$

4. $\frac{1}{18}\begin{pmatrix} 2 & 4 & 6 \\ 8 & -2 & -3 \\ 2 & 4 & -3 \end{pmatrix}$; $\langle -3, 2, 5 \rangle$, $\langle 16/9, -43/18, -13/18 \rangle$

5. $\frac{1}{9}\begin{pmatrix} -7 & 3 & 4 \\ 22 & -3 & 8 \\ 2 & 3 & -5 \end{pmatrix}$; $(10/3, -1/6, -1/6)$, $(-49/18, 145/18, 7/9)$

6. $\frac{1}{3}\begin{pmatrix} 1 & 2 & -4 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$; $(7, -4, 3)$, $(-13/3, 4, 3)$

7. $\begin{pmatrix} -i & 0 & 1+i \\ 0 & 1/3 & (1-i)/6 \\ 0 & 0 & (1-i)/2 \end{pmatrix}$; $(2i, 4, 2+3i)$, $\big(1+5i, (13+i)/6, (5+i)/2\big)$

8. $\begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1/5 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1/6 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1/7 & 0 & 0\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$; $(-2, 0, 0, 4, 0, 0, 1, 0, \cdots)$, $(-1, 0, 0, 4/5, 0, 0, 1/8, 0, \cdots)$

9.(a) $\begin{pmatrix} 1 & 0 & 1/4 & 0 & 9/64 \\ 0 & 1 & 0 & 3/8 & 0 \\ 0 & 0 & 3/4 & 0 & 5/16 \\ 0 & 0 & 0 & 5/8 & 0 \\ 0 & 0 & 0 & 0 & 35/64 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & -1/3 & 0 & -1/15 \\ 0 & 1 & 0 & -3/5 & 0 \\ 0 & 0 & 4/3 & 0 & -16/21 \\ 0 & 0 & 0 & 8/5 & 0 \\ 0 & 0 & 0 & 0 & 64/35 \end{pmatrix}$

10.(a) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 1/4 & 0 & 3/8 & 0 & 0 \\ 0 & 9/8 & 0 & 5/16 & 0 \\ 57/32 & 0 & 75/32 & 0 & 35/128 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -2/3 & 0 & 8/3 & 0 & 0 \\ 0 & -36/5 & 0 & 16/5 & 0 \\ -4/5 & 0 & -160/7 & 0 & 128/35 \end{pmatrix}$

§1.7 Subspace Components of Vectors

The coefficients v_x, v_y, and v_z in the representation $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ of vectors in $\mathcal{G}^3$ are called the **scalar components** of the vector. In previous discussions, we have simply called them components, but because we are about to define what are called *vector components* of vectors, we have added the adjective "scalar". Should we happen to omit either adjective, scalar or vector, context always makes it clear which type of component is intended.

Every vector $\mathbf{v} = v_x\hat{\mathbf{i}}+v_y\hat{\mathbf{j}}+v_z\hat{\mathbf{k}}$ in $\mathcal{G}^3$ can be regarded as the sum of the vector $v_x\hat{\mathbf{i}}$ along the x-axis, and the vector $v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ in the yz-plane. We call $v_x\hat{\mathbf{i}}$ the **vector component** of $\mathbf{v}$ along the x-axis, and $v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ the vector component of $\mathbf{v}$ in the yz-plane. Vectors along the x-axis constitute a subspace of $\mathcal{G}^3$, call it W_1; vectors in the yz-plane also form a subspace, call it W_2. In other words, every vector $\mathbf{v}$ in $\mathcal{G}^3$ can be expressed as the sum of two vectors

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2 \tag{1.14}$$

where $\mathbf{w}_1$ is in W_1 and $\mathbf{w}_2$ is in W_2, and the representation is unique. We call $\mathbf{w}_1$ the **vector component** of $\mathbf{v}$ along W_1 as determined by W_2, and $\mathbf{w}_2$ the vector component of $\mathbf{v}$ along W_2 as determined by W_1. It is important to notice that the only vector in both subspaces is the zero vector.

Every vector $(x_1, x_2, \ldots, x_n)$ in $\mathcal{R}^n$ can be expressed uniquely in the form

$$(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_i, 0, 0, \ldots, 0) + (0, 0, \ldots, 0, x_{i+1}, x_{i+2}, \ldots, x_n).$$

Vectors of the form $(x_1, x_2, \ldots, x_i, 0, 0, \ldots, 0)$ constitute a subspace of $\mathcal{R}^n$, call it W_1; vectors of the form $(0, 0, \ldots, 0, x_{i+1}, x_{i+2}, \ldots, x_n)$ constitute another subspace, call it W_2. Vector $(x_1, x_2, \ldots, x_i, 0, 0, \ldots, 0)$ is the vector component of $\mathbf{v}$ along the subspace W_1 as determined by W_2; and $(0, 0, \ldots, 0, x_{i+1}, x_{i+2}, \ldots, x_n)$ is the vector component of $\mathbf{v}$ along the subspace W_2 as determined by W_1. Once again, the only vector common to W_1 and W_2 is the zero vector.

The xy-plane and the yz-plane are subspaces of $\mathcal{G}^3$. It is possible to express every vector in $\mathcal{G}^3$ as the sum of vectors in these subspaces, but there are many ways to do it; the representation is not unique. The reason for this is that the subspace consisting af all vectors along the y-axis is common to the subspaces.

We now extend these ideas to arbitrary vector spaces.

Definition 1.7 If W_1 and W_2 are subspaces of a vector space V such that every vector $\mathbf{v}$ in V can be expressed uniquely in the form $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in W_1, and $\mathbf{w}_2$ is in W_2, we say that V is the **direct sum** of W_1 and W_2, and write that $V = W_1 \oplus W_2$. We call $\mathbf{w}_1$ the **vector component of v along** W_1 **as determined by** W_2, and $\mathbf{w}_2$ the **vector component of v along** W_2 **as determined by** W_1†.

The requirement that the representation of a vector in the form $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in W_1 and $\mathbf{w}_2$ is in W_2, is unique is tantamount to the requirement that only the zero vector is common to W_1 and W_2. We prove this in the following theorem.

Theorem 1.18 Suppose that all vectors in a vector space V can be expressed in the form $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are vectors in subspace W_1 and W_1, respectively. The representation is unique if, and only if, the only vector in the intersection of the subspaces is the zero vector.

Proof First suppose that W_1 and W_2 have only the zero vector in common, and yet there exists a vector in V that has two different representations $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{v} = \mathbf{w}_1' + \mathbf{w}_2'$. Then

† Many authors refer to these as projections of $\mathbf{v}$ onto the subspaces W_1 and W_2.

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1' + \mathbf{w}_2' \qquad \Longrightarrow \qquad \mathbf{w}_1 - \mathbf{w}_1' = \mathbf{w}_2' - \mathbf{w}_2.$$

But the left vector is in W_1 and the right one is in W_2, contradicting the fact that only the zero vector is common to the subspaces. Conversely, suppose that vectors in V can be expressed uniquely in the form $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, and yet W_1 and W_2 have at least one nonzero vector in common, call it $\mathbf{u}$. According to Exercise 40 in Section 1.2, the intersection of subspaces in a vector space is a subspace. Hence, the vector $-\mathbf{u}$ must also be in the intersection. We can therefore express the zero vector, which is in V, in two different ways in terms of vectors in W_1 and W_2, namely,

$$\mathbf{0} = -\mathbf{u} + \mathbf{u}, \qquad \text{and} \qquad \mathbf{0} = \mathbf{0} + \mathbf{0},$$

a contradiction.∎

As an immediate corollary, we have the following result.

Corollary 1.18.1 Suppose that all vectors in a vector space V can be expressed in the form $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are vectors in subspace W_1 and W_2, respectively. Then $V = W_1 \oplus W_2$ if, and only if, the intersection of the subspaces is the zero vector.

We only take vector components of a vector along one subspace as determined by another when the space is the direct sum of the subspaces. As pointed out above, we can express every vector in $\mathcal{G}^3$ as the sum of vectors in the plane xy-plane and the yz-plane, but the representation is not unique. $\mathcal{G}^3$ is not the direct sum of these two subspaces, and we do not take components of a vector along either of these subspaces as determined by the other.

It is important to notice that a vector component depends not only on the subspace where the component is to be found but also on the other subspace in the direct sum. The next two examples illustrate this point.

Example 1.29 Find the vector component of a vector $\mathbf{v} = \langle v_1, v_2 \rangle$ in $\mathcal{G}^2$ along the subspace W_1 of vectors of the form $\langle v_1, 0 \rangle$ as determined by the subspace W_2 of vectors of the form $\langle 0, v_2 \rangle$.

Solution Since $\mathbf{v} = \langle v_1, v_2 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle$, it follows that the vector component of $\mathbf{v}$ along W_1 as determined by W_2 is $\langle v_1, 0 \rangle$.•

Contrast this with the following example.

Example 1.30 Find the vector component of a vector $\mathbf{v} = \langle v_1, v_2 \rangle$ in $\mathcal{G}^2$ along the subspace W_1 of vectors of the form $\langle v_1, 0 \rangle$ as determined by the subspace W_2 of vectors that are multiples of $\langle 3, 2 \rangle$.

Solution Because vectors $\langle 1, 0 \rangle$ and $\langle 3, 2 \rangle$ are linearly independent, they constitute a basis for $\mathcal{G}^2$, and every vector $\mathbf{v}$ in the space can be expressed in the form $\mathbf{v} = c_1\langle 1, 0 \rangle + c_2\langle 3, 2 \rangle$. The vector component of $\mathbf{v}$ along W_1 as determined by W_2 is $c_1\langle 1, 0 \rangle$. To find c_1, we equate components in

$$\langle v_1, v_2 \rangle = c_1\langle 1, 0 \rangle + c_2\langle 3, 2 \rangle,$$

to get

$$v_1 = c_1 + 3c_2, \quad v_2 = 2c_2.$$

The solution is $c_1 = v_1 - 3v_2/2$ and $c_2 = v_2/2$. In other words,

$$\langle v_1, v_2 \rangle = \left(v_1 - \frac{3v_2}{2} \right) \langle 1, 0 \rangle + \frac{v_2}{2} \langle 3, 2 \rangle.$$

The vector component of $\mathbf{v}$ along W_1 as determined by W_2 is

$$\left(v_1 - \frac{3v_2}{2}\right)\langle 1, 0\rangle.$$

We have shown the vector $\mathbf{v}$ and its vector components in Figure 1.3.

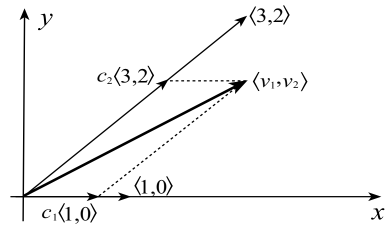

Figure 1.3

EXERCISES 1.7

1. Repeat Example 1.30 but replace the vector $\langle 3, 2\rangle$ with the vector $\langle -2, 1\rangle$. In addition, find the vector component of $\langle x_1, x_2\rangle$ along W_2 as determined by W_1.

2. Let W_1 be the subspace of $\mathcal{R}^2$ of vectors that are multiples of $(1, 4)$, and W_2 be the subspace of multiples of $(2, -3)$. Find the vector component of a vector $\mathbf{v} = (x_1, x_2)$ along W_1 as determined by W_2, and the vector component along W_2 as determined by W_1.

3. Let W_1 be the subspace of $\mathcal{G}^3$ consisting of vectors in the xy-plane, and W_2 be the subspace of vectors along the z-axis. What is the vector component of a vector $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ along W_1 as determined by W_2, and the vector component along W_2 as determined by W_1.

4. Repeat Exercise 3 but take W_2 as multiples of the vector $\langle 1, 2, 3\rangle$.

5. Let W_1 be the subspace of $\mathcal{R}^3$ spanned by the vectors $(1, 2, 3)$ and $(-2, 4, 1)$, and let W_2 be the subspace spanned by $(3, 2, 1)$. Find the vector component of a vector $\mathbf{v} = (x_1, x_2, x_3)$ along W_1 as determined by W_2, and the vector component along W_2 as determined by W_1.

6. Let W_1 be the subspace of $P_3(x)$ spanned by 1 and x, and let W_2 be the subspace spanned by x^2 and x^3. Find the vector component of a vector $p(x) = a_0 + a_1x + a_2x^2 + a^3x^3$ along W_1 as determined by W_2, and the vector component along W_2 as determined by W_1.

7. Let W_1 be the subspace of $P_3(x)$ spanned by $1 - x$ and x^2, and let W_2 be the subspace spanned by $3x$ and $2x^3$. Find the vector component of a vector $p(x) = a_0 + a_1x + a_2x^2 + a^3x^3$ along W_1 as determined by W_2, and the vector component along W_2 as determined by W_1.

8. Let W_1 be the subspace of $P_2(x)$ spanned by $1 - x + x^2$ and $2 + x - 3x^2$, and let W_2 be the subspace spanned by $4 + 2x - 5x^2$. Find the vector component of a vector $p(x) = a_0 + a_1x + a_2x^2$ along W_1 as determined by W_2, and the vector component along W_2 as determined by W_1.

9. Does the component of a vector along a subspace W_1 as determined by W_2 depend on the bases for W_1 and W_2 used to find $\mathbf{w}_1$ and $\mathbf{w}_2$?

10. Prove that $M_{n,n}(\mathcal{R})$ is the direct sum of the subspace of $n \times n$ symmetric matrices and the subspace of $n \times n$ skew-symmetric matrices. What is the dimension of each subspace?

11. Prove that the space $C^0[-a, a]$ of continuous functions on the interval $-a \le x \le a$ is the direct sum of the subspace of even, continuous functions on the interval and the subspace of odd, continuous functions on the interval.

12. In Exercises 1.2, we combined subspaces W_1 and W_2 of a vector space V to form the sum $W_1 + W_2$, the intersection $W_1 \cap W_2$, and the union $W_1 \cup W_2$.

(a) Prove that $W_1 + W_2$ contains $W_1 \cup W_2$.

(b) Is $W_1 + W_2$ the same as $W_1 \oplus W_2$?

Answers

1. $(x_1 + 2x_2)(1, 0),\ x_2(-2, 1)$ **2.** $\left(\dfrac{3x_1 + 2x_2}{11}\right)(1, 4),\ \left(\dfrac{4x_1 - x_2}{11}\right)(2, -3)$ **3.** $v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}},\ v_3\hat{\mathbf{k}}$
4. $(v_3/3)\langle 1, 1, 1\rangle,\ (v_1 - v_3/3)\hat{\mathbf{i}} + (v_2 - 2v_3/3)\hat{\mathbf{j}}$ **5.** $[-2x_1 - 5x_2 + 16x_3)/36](1, 2, 3) + [(-x_1 + 2x_2 - x_3)/9](-2, 4, 1),\ [(10x_1 + 7x_2 - 8x_3)/36](3, 2, 1)$
6. $a_0 + a_1x,\ a^2x^2 + a_3x^3$ **7.** $a_0(1 - x) + a_2x^2,\ [(a_0 + a_1)/3](3x) + (a_3/2)(2x^3)$
8. $[a_0 - 2a_1)/3](1 - x + x^2) - (a_0 + 3a_1 + 2a_2)(2 + x - 3x^2),\ [2a_0 + 5a_1 + 3a_2)/3](4 + 2x - 5x^2)$
9. No **10.** $n(n + 1)/2,\ n(n - 1)/2$ **12.**(b) Sometimes, but not always