CHAPTER 3 EIGENVALUES AND EIGENVECTORS OF LINEAR OPERATORS

§3.1 Eigenvalues and Eigenvectors

Eigenvalues and *eigenvectors* are associated with linear operators, not linear transformations in general. Linear operators map a vector space to itself, whereas linear transformations map one vector space to another. We shall see how important eigenvectors are in applications of linear algebra.

Definition 3.1 An **eigenvector** $\mathbf{v}$ of a linear operator L on a vector space V is a nonzero vector $\mathbf{v}$ that satisfies the equation

$$L(\mathbf{v}) = \lambda\,\mathbf{v}, \tag{3.1}$$

for some scalar λ. The value of λ is called the **eigenvalue** associated with the eigenvector $\mathbf{v}$. Eigenvalue and eigenvector together make an **eigenpair**.

Other terminology for eigenvalues are "latent values", "proper values", or "characteristic values".

In $\mathcal{G}^2$ and $\mathcal{G}^3$, eigenvectors corresponding to a nonzero eigenvalue are easy to visualize. They are vectors that when mapped by L, do not change direction. They may change length (due to λ), and even be in the opposite direction to $\mathbf{v}$ (if $\lambda < 0$), but they cannot be in any other direction. In any other space, $\mathbf{v}$ is an eigenvector if it is mapped to a multiple of itself by L.

In $\mathcal{G}^2$ and $\mathcal{G}^3$ wherein we can visualize vectors, the action of a linear operator may make eigenpairs clear. Three instances of this are the linear operators in Examples 2.11, 2.15, and 2.16.

Example 3.1 What are eigenpairs of the linear operator in Example 2.16?

Solution The linear operator reflects vectors in the xz-plane. Clearly any vector from the origin to a point in the xz-plane lies in the xz-plane, and its reflection in the xz-plane is itself. Hence, $L(\mathbf{v}) = \mathbf{v}$ for any such vector. Consequently, every non-zero vector that lies in the xz-plane is an eigenvector with eigenvalue $\lambda = 1$. In addition, vectors perpendicular to the xz-plane reflect to the negative of themselves. Hence, for any such vector, $L(\mathbf{v}) = -\mathbf{v}$. In other words, vectors perpendicular to the xz-plane are eigenvectors with eigenvalue $\lambda = -1$.•

Example 3.2 What are eigenpairs of the linear operator of Example 2.15 when $0 < \theta < \pi/2$?

Solution When vectors along the z-axis are rotated around the z-axis, they are unchanged. In other words, such vectors are eigenvectors with eigenvalue $\lambda = 1$.•

Example 3.3 What are eigenpairs of the linear operator in Example 2.11?

Solution The linear operator takes orthogonal components of vectors along the plane $y = x$. Clearly any vector from the origin to a point in the plane lies in the plane, and its orthogonal component is itself. Hence, $L(\mathbf{v}) = \mathbf{v}$ for any such vector. Consequently, every non-zero vector that lies in the plane $y = x$ is an eigenvector with eigenvalue $\lambda = 1$. In addition, the orthogonal component of any vector perpendicular to the plane $y = x$ is the zero vector. Hence, for any such vector, $L(\mathbf{v}) = \mathbf{0}$. In other words, vectors perpendicular to the plane are eigenvectors with eigenvalue $\lambda = 0$.•

In general, it is impossible to spot eigenvectors of linear operators as we did in the last three examples. Matrices associated with linear operators provide an alternative. An eigenvalue λ and an eigenvector $\mathbf{v}$ of a linear operator L satisfy $L(\mathbf{v}) = \lambda\,\mathbf{v}$. When a matrix A can be associated with the linear operator, then $L(\mathbf{v}) = A\,\mathbf{v}$. In such cases, we can say that $(\lambda, \mathbf{v})$ is an eigenpair for the linear operator if

$$A\mathbf{v} = \lambda\,\mathbf{v}, \quad \text{(provided } \mathbf{v} \neq \mathbf{0}). \tag{3.2}$$

In this equation, $\mathbf{v}$ must be its column vector (or matrix) representation. If we were to equate components, we would obtain a system of nonlinear equations for λ and the components of $\mathbf{v}$. Alternatively, suppose we rewrite the system in the following equivalent ways:

$$\begin{aligned} A\,\mathbf{v} &= \lambda I\mathbf{v} && (I \text{ an identity matrix}) \\ A\,\mathbf{v} - \lambda I\mathbf{v} &= \mathbf{0} && (\mathbf{0} \text{ a zero matrix}) \\ (A - \lambda I)\mathbf{v} &= \mathbf{0} \end{aligned} \tag{3.3}$$

We can regard this as a system of linear, homogeneous equations in the components of $\mathbf{v}$ (where some of the coefficients involve the unknown λ). Such a system has nontrivial solutions if and only if the determinant of the coefficient matrix is zero,

$$\det(A - \lambda I) = 0.$$

We have just established the following theorem.

Theorem 3.1 When A is the matrix associated with a linear operator L, λ is an eigenvalue of L if, and only if, it satisfies the equation

$$\det(A - \lambda I) = 0. \tag{3.4}$$

Equation 3.4 is called the **characteristic equation** of the matrix A representing the linear operator; its left side is called the **characteristic polynomial** of matrix A. This certainly would be appropriate terminology were we to call λ a "characteristic" value. Characteristic polynomial and characteristic equation are the accepted terminology no matter what term is used for eigenvalues.

Before making some general observations, we illustrate how the characteristic equation gives eigenvalues in the earlier examples of this section. The matrix associated with the linear operator of Example 2.16 is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Its eigenvalues are defined by the characteristic equation

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = -(1-\lambda)^2(1+\lambda).$$

Solutions are $\lambda = \pm 1$. These were the eigenvalues found in Example 3.1.

The matrix associated with the linear operator of Example 2.15 is

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Eigenvalues are defined by the characteristic equation

$$0 = \det\,(A - \lambda I) = \det \begin{pmatrix} \cos\theta-\lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)[(\cos\theta-\lambda)^2 + \sin^2\theta].$$

Clearly $\lambda = 1$ is an eigenvalue. In addition, we should set

$$0 = (\cos\theta - \lambda)^2 + \sin^2\theta = \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta = \lambda^2 - 2\lambda\cos\theta + 1.$$

Solutions of this quadratic are

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta},$$

an impossibility. Thus, $(\cos\theta - \lambda)^2 + \sin^2\theta \neq 0$, and $\lambda = 1$ is the only eigenvalue, as was discovered in Example 3.2.

The matrix associated with the linear operator of Example 3.3 is

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Its eigenvalues are defined by the characteristic equation

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 1/2 - \lambda & 1/2 & 0 \\ 1/2 & 1/2 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix} = -\lambda(1-\lambda)^2.$$

Solutions are $\lambda = 0$ and $\lambda = 1$. These were the eigenvalues found in Example 3.3.

Here is another example.

Example 3.4 If the matrix associated with a linear operator on some 3-dimensional space is

$$A = \begin{pmatrix} 5 & -3 & -3 \\ -6 & 8 & 6 \\ 12 & -12 & -10 \end{pmatrix},$$

find eigenvalues of the operator.

Solution Eigenvalues are given by the characteristic equation

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 5 - \lambda & -3 & -3 \\ -6 & 8 - \lambda & 6 \\ 12 & -12 & -10 - \lambda \end{pmatrix}.$$

If we expand along the first row,

$$\begin{aligned} 0 &= (5-\lambda)[(8-\lambda)(-10-\lambda) + 72] + 3[-6(-10-\lambda) - 72] - 3[72 - 12(8-\lambda)] \\ &= (5-\lambda)(\lambda^2 + 2\lambda - 8) + 3(6\lambda - 12) - 3(12\lambda - 24) \\ &= -\lambda^3 + 3\lambda^2 - 4 = -(\lambda+1)(\lambda-2)^2. \end{aligned}$$

Eigenvalues are therefore $\lambda = -1$ and $\lambda = 2$.•

These examples indicate that characteristic equation 3.4 for a linear operator on an n-dimensional space is an n^{th} degree polynomial equation in λ. Its solutions are the eigenvalues of the linear operator.

We now turn our attention to finding eigenvectors once eigenvalues have been calculated. With λ known, equation 3.3 represents a system of homogeneous, linear equations in the components of the corresponding eigenvector $\mathbf{v}$. Because the determinant of the system is zero, there is an infinity of solutions. In other words, eigenvectors are not unique; there is always an infinity of eigenvectors corresponding to every eigenvalue. Because the zero vector is never considered an eigenvector of a linear operator, the totality of eigenvectors corresponding to a single eigenvalue do not constitute a subspace of the vector space on which the operator operates. If we include the zero vector in the set of eigenvectors, the set becomes a subspace, called the **eigenspace** associated with the eigenvalue. However, realize that all nonzero vectors in the eigenspace are eigenvectors, but the zero vector is not an eigenvector.

Example 3.5 Find eigenvectors corresponding to the eigenvalues $\lambda = -1$ and $\lambda = 2$ in Example 3.4. Assume that the operator associated with the matrix operates on $\mathcal{G}^3$.

Solution Eigenvectors corresponding to $\lambda = -1$ satisfy

$$\mathbf{0} = (A + I)\mathbf{v} = \begin{pmatrix} 6 & -3 & -3 \\ -6 & 9 & 6 \\ 12 & -12 & -9 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The reduced row echelon form for the augmented matrix of this system is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

When we convert to equations,

$$v_1 - \frac{v_3}{4} = 0, \qquad v_2 + \frac{v_3}{2} = 0.$$

Eigenvectors are therefore $\mathbf{v} = \begin{pmatrix} v_3/4 \\ -v_3/2 \\ v_3 \end{pmatrix} = \dfrac{v_3}{4} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$. In other words, any multiple of $\mathbf{v} = \langle 1, -2, 4 \rangle$ is an eigenvector corresponding to $\lambda = -1$; the eigenspace of $\lambda = -1$ is a line of eigenvectors. To check our calculations, we could multiply,

$$A\,\mathbf{v} = \begin{pmatrix} 5 & -3 & -3 \\ -6 & 8 & 6 \\ 12 & -12 & -10 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \lambda\,\mathbf{v}.$$

Eigenvectors corresponding to $\lambda = 2$ are given by

$$\mathbf{0} = (A - 2I)\mathbf{v} = \begin{pmatrix} 3 & -3 & -3 \\ -6 & 6 & 6 \\ 12 & -12 & -12 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The reduced row echelon form for the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus, $v_1 - v_2 - v_3 = 0$, or, $v_1 = v_2 + v_3$. Eigenvectors are

$$\mathbf{v} = \begin{pmatrix} v_2 + v_3 \\ v_2 \\ v_3 \end{pmatrix} = v_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

In this case we have a 2-parameter family of eigenvectors. Both $\langle 1, 1, 0 \rangle$ and $\langle 1, 0, 1 \rangle$ are eigenvectors, and every linear combination of them is also an eigenvector; that is, the eigenspace of $\lambda = 2$ is a plane spanned by the eigenvectors $\langle 1, 1, 0 \rangle$ and $\langle 1, 0, 1 \rangle$. It is a 2-dimensional subspace of $\mathcal{G}^3$. We could again check our calculations by showing that for any vector in the plane, $A\,\mathbf{v} = 2\mathbf{v}$.●

Here is a more abstract example.

Example 3.6 Let L be the linear operator that maps polynomials $a + bx + cx^2 + dx^3$ in the space $P_3(x)$ according to

$$L(a + bx + cx^2 + dx^3) = a + cx^2.$$

Find all eigenpairs of the operator.

Solution If we choose the natural basis $\{1, x, x^2, x^3\}$ for the space, then the matrix associated with the operator is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Eigenvalues are given by

$$0 = \det \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = \lambda^2(1-\lambda)^2.$$

Hence, $\lambda = 0$ and $\lambda = 1$. Eigenvectors corresponding to $\lambda = 0$ have components satisfying

$$\mathbf{0} = (A - \lambda I)\mathbf{v} = A\,\mathbf{v} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

This implies that $v_1 = 0$ and $v_3 = 0$. Hence, eigenvectors corresponding to $\lambda = 0$ are

$$\mathbf{v} = (0, v_2, 0, v_4) = v_2 x + v_4 x^3.$$

A similar calculation shows that eigenvectors corresponding to $\lambda = 1$ are

$$\mathbf{v} = (v_1, 0, v_3, 0) = v_1 + v_3 x^2.\bullet$$

We cannot always use the characteristic equation to find eigenvalues of a linear operator. This is especially true when the vector space is infinite-dimensional. In such cases, we may also have a continuous set of eigenvalues, rather than a discrete set. Here is an example to illustrate.

Example 3.7 Find all real eigenpairs of the linear operator on $C^\infty(-\infty, \infty)$ that takes second derivatives of functions.

Solution Eigenvalues λ and eigenvectors $f(x)$ must satisfy the equation

$$f''(x) = \lambda f(x).$$

The auxiliary equation for this linear second differential equation is $m^2 - \lambda = 0$. When $\lambda > 0$, solutions are $m = \pm\sqrt{\lambda}$, and a general solution of the differential equation is $f(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$. In other words, every positive number λ is an eigenvalue with eigenvectors $e^{\pm\sqrt{\lambda}x}$. When $\lambda < 0$, solutions of the auxiliary equation are $m = \pm\sqrt{\lambda}i$, with corresponding solutions $c_1 \sin\sqrt{-\lambda}x + c_2 \cos\sqrt{-\lambda}x$ of the differential equation. Thus, every negative number λ is an eigenvalue with eigenvectors $\sin\sqrt{-\lambda}x$ and $\cos\sqrt{-\lambda}x$. Finally, $\lambda = 0$ is also an eigenvalue with eigenvectors 1 and x.•

For a finite-dimensional vector space, eigenvalues of a linear operator are solutions of a polynomial equation, the characteristic equation. If the vector space is complex, then the characteristic equation is complex, and it is likely that eigenvalues will be complex numbers and corresponding eigenvectors will also be complex. If the space is real, then the characteristic equation is real and eigenvalues could be real, but they might also be complex (in complex conjugate pairs). In the event that eigenvalues are complex, eigenvectors will be complex (in complex conjugate pairs). When this happens, eigenvectors are not in the original real space; they are in its complexification. We illustrate in the following example.

Example 3.8 Find all eigenpairs of the linear operator with matrix $\begin{pmatrix} 2 & 0 & -4 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{pmatrix}$.

Solution Eigenvalues are given by

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 0 & -4 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -2-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda)(-2-\lambda) - 4(-2)(1-\lambda)$$

$$= (1-\lambda)[(2-\lambda)(-2-\lambda)+8] = (1-\lambda)(\lambda^2+4).$$

Eigenvalues are therefore $\lambda = 1$, $\pm 2i$. Eigenvectors corresponding to $\lambda = 1$ satisfy

$$\mathbf{0} = (A - I)\mathbf{v} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 0 & 0 \\ 2 & 0 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The reduced row echelon form for the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus, $v_1 = 0$ and $v_3 = 0$. Eigenvectors are $\mathbf{v} = \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} = v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Eigenvectors corresponding to $\lambda = 2i$ satisfy

$$\mathbf{0} = (A - 2iI)\mathbf{v} = \begin{pmatrix} 2-2i & 0 & -4 \\ 0 & 1-2i & 0 \\ 2 & 0 & -2-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The reduced row echelon form the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1-i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus, $v_1 + (-1-i)v_3 = 0$, $v_2 = 0$. Eigenvectors are

$$\mathbf{v} = \begin{pmatrix} (1+i)v_3 \\ 0 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} 1+i \\ 0 \\ 1 \end{pmatrix}.$$

Eigenvectors corresponding to $\lambda = -2i$ are $v_3 \begin{pmatrix} 1-i \\ 0 \\ 1 \end{pmatrix}$, the complex conjugate of eigenvectors corresponding to $\lambda = 2i$.•

Eigenvalues and eigenvectors are properties of linear operators L, but they are often calculated using the matrix A associated with the operator. As a result, we sometimes say that we are finding eigenvalues and eigenvectors of matrix A. But we mean eigenvalues and eigenvectors of the linear operator associated with the matrix.

When $\mathbf{v}$ is an eigenvector of a linear operator corresponding to eigenvalue λ, then $c\,\mathbf{v}$ is also an eigenvector corresponding to the same eigenvalue for any nonzero constant c. In other words, eigenvectors are only determined to a scale factor. It will be our practice to drop the constant, and simply say that an eigenvector corresponding to λ is $\mathbf{v}$. Likewise, as in Example 3.5, where any linear combination of the eigenvectors $\langle 1,1,0\rangle$ and $\langle 1,0,1\rangle$ is also an eigenvector, we might only say that $\langle 1,1,0\rangle$ and $\langle 1,0,1\rangle$ are eigenvectors of the operator corresponding to eigenvalue $\lambda = 2$ (and understand that any linear combination is also an eigenvector).

Much of what is to follow in these notes is how to use eigenvectors to simplify the solution of complicated problems, and this will depend on both the number of eigenvectors and their properties. Essentially, we know everything about eigenvalues of a linear operator. An operator on an n-dimensional space has n eigenvalues; they need not all be distinct, and they can be real or complex. Questions that we will ask about eigenvectors include:

(1) Are eigenvectors corresponding to different eigenvalues linearly independent?

(2) If an eigenvalue has multiplicity $k > 1$, how many linearly independent eigenvectors can we expect to be associated with the eigenvalue?

The first question is answered in the following theorem; the next section deals with the more complicated second question.

Theorem 3.2 Eigenvectors corresponding to different eigenvalues are linearly independent.

Proof Let $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ be eigenpairs of a linear operator L corresponding to distinct eigenvalues λ_1 and λ_2. Suppose that the eigenvectors are linearly dependent so that one is a multiple of the other $\mathbf{v}_2 = a\mathbf{v}_1$. Then,

$$L(\mathbf{v}_2) = \lambda_2 \mathbf{v}_2 = a\lambda_2 \mathbf{v}_1 \qquad \text{and} \qquad L(\mathbf{v}_2) = L(a\mathbf{v}_1) = a\lambda_1 \mathbf{v}_1.$$

These imply that $\lambda_1 = \lambda_2$, a contradiction.∎

An interesting and sometimes useful theoretical result is that the matrix A of a linear operator satisfies its characteristic equation.

Theorem 3.3 (**Cayley-Hamilton Theorem**) If $p(\lambda) = 0$ is the characteristic equation of a linear operator with matrix A, then

$$p(A) = 0. \tag{3.5}$$

To be more specific, if

$$(-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0, \tag{3.6a}$$

is the characteristic equation for a linear operator on an n-dimensional space, then

$$(-1)^n A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I = \mathbf{0}, \tag{3.6b}$$

where A is the matrix associated with the operator, I is the $n \times n$ identity matrix, and $\mathbf{0}$ is the $n \times 1$ zero matrix.

EXERCISES 3.1

In Exercises 1–13 find all eigenvalues and corresponding eigenvectors of a linear operator associated with the given matrix.

1. $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$

2. $\begin{pmatrix} 5 & 2 & 2 \\ 3 & 6 & 3 \\ 6 & 6 & 9 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & -3 & 0 \end{pmatrix}$

4. $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

5. $\begin{pmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{pmatrix}$

6. $\begin{pmatrix} 7 & 4 & -16 \\ 2 & 5 & -8 \\ 2 & 2 & -5 \end{pmatrix}$

7. $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

8. $\begin{pmatrix} 4 & 2 & -2 & 2 \\ 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & -3 & 5 \end{pmatrix}$

9. $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 2 \end{pmatrix}$

11. $\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & -i \end{pmatrix}$

12. $\begin{pmatrix} 5 & 3 & 3 & 0 & 4 \\ -3 & -5 & -2 & 7 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix}$

13. $\begin{pmatrix} 1-i & 1 & 0 & 0 & 0 \\ 0 & 1-i & 0 & 0 & 0 \\ i & -1 & 1 & 0 & 0 \\ 0 & i & 1 & 1 & 1 \\ -i & 1 & 0 & 0 & 1 \end{pmatrix}$

14. Prove that $\lambda = 0$ is an eigenvalue of a matrix A if, and only if, A is singular.

15. Prove that if $(\lambda, \mathbf{v})$ is an eigenpair of a matrix A which has an inverse, then $(\lambda^{-1}, \mathbf{v})$ is an eigenpair for A^{-1}.

16. (a) Prove that if $(\lambda, \mathbf{v})$ is an eigenpair of a real matrix A, then $(\overline{\lambda}, \overline{\mathbf{v}})$ is also an eigenpair.

(b) Use the result in part (a) to aid in finding all eigenpairs of the matrix $\begin{pmatrix} -5 & 6 & 2 \\ -3 & 4 & 1 \\ -5 & 5 & 2 \end{pmatrix}$.

17. Show that if $(\lambda, \mathbf{v})$ is an eigenpair for A, then $(\lambda^n, \mathbf{v})$ is an eigenpair for A^n.

18. A square matrix is said to be **idempotent** if $A^2 = A$. Show that $\lambda = 0$ and $\lambda = 1$ are the only eigenvalues for such a matrix.

19. Find all eigenpairs of the matrix $\begin{pmatrix} 1 & 2 & 18 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

20. Let L be the linear operator on the space $P_1(x)$ defined by

$$L(a + bx) = b + 4ax.$$

Find all eigenpairs of the operator.

21. Let L be the linear operator on the space $P_2(x)$ defined by

$$L(c + bx + ax^2) = c + ax + bx^2.$$

Find all eigenpairs of the operator.

22. Repeat Exercise 21 for the operator

$$L(c + bx + ax^2) = -b + ax + cx^2.$$

23. Let L be the linear operator on $C^\infty(-\infty, \infty)$ that maps functions (vectors) according to $L\big(f(x)\big) = f'(x)$. Find all real eigenpairs of the operator.

24. Let L be the linear operator on the space $P(x)$ that maps polynomials according to $L\big(p(x)\big) = xp'(x)$, where $p'(x)$ is the derivative of $p(x)$. Find eigenpairs of the operator.

25. Let L be the linear operator on $C^\infty(-\infty,\infty)$ that maps functions (vectors) according to $L\big(f(x)\big) = f''(x) + f'(x)$. Find all real eigenpairs of the operator.

26. Find a 3×3 matrix whose eigenvalues are 0, 1, and -1, with corresponding eigenvectors $(1,1,0)$, $(1,1,1)$, and $(1,0,1)$, respectively.

27. Find a 3×3 matrix whose eigenvalues are 1 and $(3\pm\sqrt{7}i)/2$, with corresponding eigenvectors $(-1,1,0)$, and $(1\mp\sqrt{7}i,-2,2)$, respectively.

28. Suppose A is an $n\times n$ matrix with eigenpair $(\lambda,\mathbf{v})$. Show that if P is an $n\times n$ invertible matrix, then the matrix PAP^{-1} has eigenpair $(\lambda, P\mathbf{v})$.

29. Show that the eigenvalues of the transpose A^T of a square matrix A are the same as those for A. Are eigenvectors for A^T the same as those for A?

30. A square matrix is said to be **nilpotent** if for some integer $m\ge 2$, A^m is the zero matrix. Show that $\lambda = 0$ is the only eigenvalue for such a matrix.

31. Is the eigenspace of an eigenvalue λ of a matrix A the kernel of the operator $A-\lambda I$?

32. Does the characteristic polynomial of the matrix associated with a linear transformation depend on the basis used to find the matrix?

33. Find necessary and sufficient conditions on constants a, b, and c in order that the matrix

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{pmatrix}$$

be diagonalizable.

34. Given that $a>0$ and b, c, p, and q are nonnegative constants, show that the matrix

$$\begin{pmatrix} a & b & c \\ p & 0 & 0 \\ 0 & q & 0 \end{pmatrix}$$

has exactly one positive real eigenvalue.

Answers

1. $\lambda = 5$, $(1,0,0)$
2. $\lambda = 3$, $(-1,1,0)$, $(-1,0,1)$; $\lambda = 14$, $(2,3,6)$
3. $\lambda = -1$, $(-2,1,3)$; $\lambda = 1$, $(1,0,0)$; $\lambda = 3$, $(0,-1,1)$
4. $\lambda = -1$, $(-1,1,0)$, $(-1,0,1)$; $\lambda = 2$, $(1,1,1)$
5. $\lambda = 1$, $(1,1,1)$; $\lambda = 2$, $(2,1,1)$; $\lambda = 3$, $(1,1,2)$
6. $\lambda = 1$, $(2,1,1)$; $\lambda = 3$, $(-1,1,0)$, $(4,0,1)$
7. $\lambda = 1$, $(0,1,0,0)$, $(0,0,1,0)$; $\lambda = 2$, $(0,0,2,1)$; $\lambda = 3$, $(1,2,0,0)$
8. $\lambda = 2$, $(1,-1,0,0)$, $(0,0,1,1)$; $\lambda = 4$, $(0,-1,0,1)$; $\lambda = 6$, $(1,0,0,1)$
9. $\lambda = 0$, $(1,i)$; $\lambda = 1$, $(i,1)$
10. $\lambda = 2$, $(0,1,0)$; $\lambda = (3+\sqrt{5})/2$, $\big(-i(\sqrt{5}-1),0,2\big)$; $\lambda = (3-\sqrt{5})/2$, $\big(i(\sqrt{5}+1),0,2\big)$
11. $\lambda = i$, $(1,i,0,0)$, $(0,0,1,0)$; $\lambda = -i$, $(i,1,0,0)$
12. $\lambda = -4$, $(-1,3,0,0,0)$; $\lambda = 4$,$(-3,1,0,0,0)$; $\lambda = 3$, $(-67,19,7,-7,14)$; $\lambda = 2$, $(-1,-6,3,-6,3)$; $\lambda = 1$, $(-3,9,-5,5,0)$
13. $\lambda = 1$, $(0,0,0,1,0)$, $(0,0,1,0,-1)$; $\lambda = 1-i$, $(1,0,-1,0,1)$
16.(b) $\lambda = 1$, $(1,1,0)$; $\lambda = i$, $(2,1,2+i)$; $\lambda = -i$, $(2,1,2-i)$
19. $\lambda = 5$, $(6,3,1)$; $\lambda = (-1+\sqrt{23}i)/2$, $(2+2\sqrt{23}i, 5-\sqrt{23}i, 2)$; $\lambda = (-1-\sqrt{23}i)/2$, $(2-2\sqrt{23}i, 5+\sqrt{23}i, 2)$;
20. $\lambda = 2$, $1+2x$; $\lambda = -2$, $1-2x$
21. $\lambda = 1$, 1, $x+x^2$; $\lambda = -1$, $x-x^2$
22. $\lambda = -1$, $-1+x+x^2$; $\lambda = (1+\sqrt{3}i)/2$, $(1+\sqrt{3}i)+(1-\sqrt{3}i)x+2x^2$; $\lambda = (1-\sqrt{3}i)/2$, $(1-\sqrt{3}i)+(1+\sqrt{3}i)x+2x^2$

23. λ any real number, $e^{\lambda x}$
24. $\lambda =$ a nonnegative integer, $p(x) = x^\lambda$
25. $\lambda = -1/4$, $e^{-x/2}$, $xe^{-x/2}$; $\lambda > -1/4$, $e^{(-1\pm\sqrt{1+4\lambda}x/2}$;
$\lambda < -1/4$, $e^{-x/2}\cos\dfrac{\sqrt{-1-4\lambda}x}{2}$, $e^{-x/2}\sin\dfrac{\sqrt{-1-4\lambda}x}{2}$

26. $\begin{pmatrix} -2 & 2 & 1 \\ -1 & 1 & 1 \\ -2 & 2 & 1 \end{pmatrix}$ **27.** $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}$ **29.** Not always **31.** Yes **32.** No **33.** $a = 0$

§3.2 Bases of Eigenvectors

In this section we show that when a linear operator L operates on a vector space V, and there are a sufficient number of eigenvectors to make a basis for the vector space, the matrix of the operator takes on a particularly simple form. We illustrate with the linear transformation on $\mathcal{G}^3$ in Examples 3.4 and 3.5. In these examples, we found that $\lambda = 1$ is an eigenvalue with corresponding eigenvector $\langle 1, -2, 4\rangle$, and that $\lambda = 2$ is also an eigenvalue with two linearly independent eigenvectors $\langle 1, 1, 0\rangle$ and $\langle 1, 0, 1\rangle$. It is straightforward to show that all three eigenvectors are linearly independent, and we could therefore use them as a basis for $\mathcal{G}^3$. If we do this, then the matrix associated with the operator is diagonal. To see this, recall that the columns of the matrix are the transforms of the basis vectors. But,

$$L\big(\langle 1, -2, 4\rangle\big) = \langle 1, -2, 4\rangle, \quad L\big(\langle 1, 1, 0\rangle\big) = 2\langle 1, 1, 0\rangle, \quad L\big(\langle 1, 0, 1\rangle\big) = 2\langle 1, 0, 1\rangle.$$

In other words, the matrix associated with the operator relative to the basis $\langle 1, -2, 4\rangle$, $\langle 1, 1, 0\rangle$, and $\langle 1, 0, 1\rangle$ is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We say that the matrix associated with the linear operator has been **diagonalized**. Diagonalization is possible in this example because there are three linearly independent eigenvectors of the linear operator. When the number of linearly independent eigenvectors is equal to the dimension of the space, we have the following theorem.

Theorem 3.4 If a linear operator on a real, n-dimensional vector space has n real, linearly independent eigenvectors, and they are used as a basis for the space, then the matrix of the linear operator is a diagonal matrix with eigenvalues on the diagonal.

When this theorem is combined with Theorem 3.2, we obtain the following corollary.

Corollary 3.4.1 If a linear operator on a real, n-dimensional vector space has n real, distinct eigenvalues, and corresponding eigenvectors are used as a basis for the space, then the matrix of the linear operator is a diagonal matrix with eigenvalues on the diagonal.

In Section 3.1, we saw that an eigenvalue can have multiplicity larger than one. We call this the **algebraic multiplicity** of the eigenvalue. In some cases, the number of linearly independent eigenvectors for such an eigenvalue may turn out to be less than the algebraic multiplicity of the eigenvalue, in which case the total number of linearly independent eigenvectors for all eigenvalues is less than the dimension of the space. We call the number of linearly independent eigenvectors associated with an eigenvalue the **geometric multiplicity** of the eigenvalue. Here are two examples to illustrate.

Example 3.9 Find eigenpairs for the linear operator with matrix $A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{pmatrix}$. In particular, confirm that the eigenvalue with algebraic multiplicity 2 has geometric multiplicity 1.

Solution Eigenvalues are defined by the characteristic equation

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & 2 \\ -2 & 0 & 3-\lambda \end{pmatrix} = -\lambda^3 + 5\lambda^2 - 7\lambda + 3$$
$$= -(\lambda - 1)^2(\lambda - 3).$$

Eigenvalues are $\lambda = 3$, with (algebraic) multiplicity 1, and $\lambda = 1$, with (algebraic) multiplicity 2. Eigenvectors corresponding to $\lambda = 3$ are multiples of $(0, 1, 1)$, and those corresponding to $\lambda = 1$ are multiples of $(1, 1, 1)$. Thus, the geomtric multiplicity of $\lambda = 1$ is 1.•

Example 3.10 Show that the linear operator with matrix $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has only a single eigenvalue with algebraic multiplicity 3 and geometric multiplicity 2.

Solution Eigenvalues are defined by the characteristic equation

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^3.$$

There is only one eigenvalue $\lambda = 2$ with algebraic multiplicity 3. Eigenvectors are given by

$$\mathbf{0} = (A - 2I)\mathbf{v} = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Eigenvectors are

$$\begin{pmatrix} v_1 \\ -4v_3/3 \\ v_3 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{v_3}{3} \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}.$$

Thus, there are two linearly independent eigenvectors $(1, 0, 0)$ and $(0, -4, 3)$; that is, the eigenvalue has geometric multiplciity 2.•

When the number of linearly independent eigenvectors of a linear operator is less than the dimension n of the space, the matrix associated with the linear operator cannot be diagonalized. Obviously, we should now determine when a linear operator has n linearly independent eigenvectors, and what, if anything, we can do to simplify the matrix associated with the operator in the event that there is less than n linearly independent eigenvectors. The following theorem is clear.

Theorem 3.5 The matrix of a linear operator on a real vector space is diagonalizable if all eigenvalues are real, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.

The linear operators in Examples 3.9 and 3.10 fail to meet the criterion of this theorem.

Theorems 3.4 and 3.5, and the corollary to Theorem 3.4 indicate that the matrix of a linear operator on a real vector space is diagonalizable only if its eigenvalues are real. These results are also valid for complex vector spaces in which case eigenvalues and eigenvectors are complex, and the resulting diagonal matrix will have complex entries. When an operator on a real vector space has complex values, the matrix of the operator is not diagonalizable in the real space, but it is diagonalizable in the complexification of the space.

EXERCISES 3.2

For Exercises 1–8 determine whether the matrices in Exercise 1–8 in Section 3.1 are diagonalizable. If they are, find the matrix for the linear transformation when the eigenvectors are used as a basis for the space.

Answers

1. Not diagonalizable **2.** $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$ **3.** $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ **4.** $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ **6.** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ **7.** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ **8.** $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$

§3.3 Generalized Eigenvectors and Jordan Canonical Form

In Section 3.2, we saw that the matrix of a linear operator is diagonalizable if the number of linearly independent eigenvectors of the operator is equal to the dimension of the space. *Generalized eigenvectors* of linear operators can be defined and used to achieve a simplified form for the matrix when the number of eigenvectors is less than the dimension of the space, but the matrix will not be diagonal. Discussion of this topic is beyond the scope of these notes, but we would at least like the reader to be aware of what can be achieved.

Suppose that the characteristic equation of the matrix A associated with a linear operator on a 5-dimensional space is $p(\lambda) = (\lambda+9)^3(\lambda-5)^2$, with therefore eigenvalues $\lambda = -9$ and $\lambda = 5$, with algebraic multiplicities 3 and 2, respectively. Suppose that geometric multiplicities are 1 and 2, so that there are only three linearly independent eigenvectors, one associated with $\lambda = -9$, and two with $\lambda = 5$. The Jordan canoncial form for the matrix is

$$\begin{pmatrix} -9 & 1 & 0 & 0 & 0 \\ 0 & -9 & 1 & 0 & 0 \\ 0 & 0 & -9 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

The top 3×3 block has eigenvalue 9 on the diagonal, and 1's above the diagonal. We could consider the bottom 2×2 block has eigenvalue 5 on the diagonal and no 1's above the diagonal. It is better to consider it has two 1×1 blocks with 5 on the diagonal.

If the characteristic equation is $p(\lambda) = (\lambda-2)^4(\lambda-5)^2$ and geometric multiplicities of $\lambda = 2$ and $\lambda = 5$ are 2 and 1, respectively, the Jordan canonical form for the matrix associated with the linear operator is

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \qquad \text{or,} \qquad \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

Both have three blocks. In the first, the top 3×3 block has eigenvalue 2 on the diagonal, 1's above the diagonal. The middle block is a 1×1 matrix with entry 2. The bottom 2×2 block has eigenvalue 5 on the diagonal and 1 above the diagonal. The second form has two 2×2 blocks each with 2 on the diagonal and a 1 above the diagonal. The bottom block is the same.

If the characteristic equation is $p(\lambda) = (\lambda-3)^7$, and the geometric multiplicity of $\lambda = 3$ is 3, there are three possible Jordan canonical forms for the matrix associated with the linear operator

$$\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Each has three blocks. In the first form, the top 5×5 block has 4's on the diagonal and 1' above. In addition, there are two 1×1 blocks with 4 on the diagonal. In the second form, there is a 4×4 block, a 2×2 block, and a 1×1 block. Each has 4' on the diagonal and 1 above. In the third form, there is two 3×3 blocks and a 1×1 block with 4' on the diagonal and 1's above.

Here is the theorem that describes the Jordan canonical form in general.

Theorem 3.6 If L is an operator on a finite dimensional vector space, then there exists a basis of the space such that the matrix of the operator is block diagonal with each block of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots 0 & 0 & \lambda \end{pmatrix}.$$

All entries outside the blocks are 0. The number of blocks corresponding to an eigenvalue is the geometric multiplicity of the eigenvalue. The algebraic multiplicity of an eigenvalue is the total dimension of all blocks containing the eigenvalue.