CHAPTER 4 APPLICATIONS OF LINEAR ALGEBRA

In this chapter we explore various mathematical problems, solutions of which can be simplified with what we have learned about linear algebra.

§4.1 Linear, First-order Differential Equations

In Section 4.2, we discuss systems of coupled, linear, first-order differential equations. By coupled we mean that if we have two differential equations in two unknowns, say $x(t)$ and $y(t)$, then both equations contain both unknowns. We shall use linear algebra to decouple the equations so that each equation contains only one of the unknowns. It will then be a matter of solving two linear, first-order differential equations each with one unknown. A linear, first-order differential equation in $y(t)$ is one that can be expressed in the form

$$\frac{dy}{dt} + P(t)y = Q(t). \tag{4.1}$$

Such equations are solved by introducing an integrating factor

$$e^{\int P(t)\,dt}. \tag{4.2}$$

When each term in the differential equation is multiplied by this factor, the result is

$$e^{\int P(t)\,dt}\frac{dy}{dt} + P(t)ye^{\int P(t)\,dt} = Q(t)e^{\int P(t)\,dt}.$$

But the left side of the differential equation is now the derivative of a product,

$$\frac{d}{dt}\left[ye^{\int P(t)\,dt}\right] = Q(t)e^{\int P(t)\,dt}.$$

We can integrate both sides of the equation to get

$$ye^{\int P(t)\,dt} = \int Q(t)e^{\int P(t)\,dt}dt + C,$$

where C is a constant of integration. Division by the integrating factor gives a general solution of the differential equation

$$y(t) = e^{-\int P(t)\,dt}\left[\int Q(t)e^{\int P(t)\,dt}dt\right] + Ce^{-\int P(t)\,dt}. \tag{4.3}$$

In saying that expression 4.3 is a general solution of the differential equation, we mean that it contains all solutions of the equation. Here are some examples to illustrate.

Example 4.1 Find a general solution of the differential equation $\dfrac{dy}{dt} + ty = t$.

Solution An integrating factor for this linear equation is

$$e^{\int t\,dt} = e^{t^2/2}.$$

If we multiply each term in the equation by this factor, we have

$$e^{t^2/2}\frac{dy}{dt} + yte^{t^2/2} = te^{t^2/2},$$

and we know that the left side must be the derivative of the product of y and $e^{t^2/2}$,

$$\frac{d}{dt}\left(ye^{t^2/2}\right) = te^{t^2/2}.$$

(It is always a good idea to do the differentiation on the left to verify that the integrating factor is indeed correct.) Integration yields

$$ye^{t^2/2} = \int te^{t^2/2}dt = e^{t^2/2} + C \qquad \Longrightarrow \qquad y(t) = 1 + Ce^{-t^2/2}.$$

This solution is valid for all values of t.●

Example 4.2 Find a general solution for the differential equation $\quad t\dfrac{dy}{dt} + y - t\,\sin t = 0.$

Solution If we write the differential equation in the form

$$\frac{dy}{dt} + \frac{y}{t} = \sin t,$$

we see that it is linear first-order. An integrating factor is therefore

$$e^{\int (1/t)\,dt} = e^{\ln|t|} = |t|.$$

If we multiply each term in the differential equation by this factor, we get

$$|t|\frac{dy}{dt} + \frac{|t|}{t}y = |t|\,\sin t.$$

When $t > 0$, the absolute values may be discarded

$$t\frac{dy}{dt} + \frac{t}{t}y = t\,\sin t,$$

whereas when $t < 0$, they are replaced by negative signs,

$$-t\frac{dy}{dt} - \frac{t}{t}y = -t\,\sin t.$$

In either case, however, the equation simplifies to

$$t\frac{dy}{dt} + y = t\,\sin t.$$

Notice that this is exactly the form in which we found the differential equation. In other words, had we paused for a moment at the outset, we might have realized that the combination of terms $t\,dy/dt + y$ in the original differential equation is the derivative of the product ty. Having failed to do this, the above analysis confirms that this is indeed the case and the differential equation can be expressed in the form

$$\frac{d}{dt}(ty) = t\,\sin t.$$

Integration now gives

$$ty = \int t\,\sin t\,dt = -t\,\cos t + \sin t + C.$$

The explicit solution is

$$y(t) = -\cos t + \frac{\sin t}{t} + \frac{C}{t}.$$

Nonexistence of the solution at $t = 0$ suggests the more detailed description

$$y(t) = \begin{cases} -\cos t + \dfrac{\sin t}{t} + \dfrac{C_1}{t}, & t < 0 \\ -\cos t + \dfrac{\sin t}{t} + \dfrac{C_2}{t}, & t > 0. \end{cases}$$●

EXERCISES 4.1

In Exercises 1–12 find a general solution for the differential equation.

1. $\dfrac{dy}{dt} + 2ty = 4t$

2. $\dfrac{dy}{dt} + \dfrac{2}{t}y = 6t^3$

3. $(2y - t) + \dfrac{dy}{dt} = 0$

4. $\dfrac{dy}{dt} + y \cot t = 5e^{\cos t}$

5. $(t^2 + 1)\dfrac{dy}{dt} = -(t^2 + 2ty)$

6. $(t + 1)\dfrac{dy}{dt} - 2y = 2(t + 1)$

7. $\dfrac{1}{t}\dfrac{dy}{dt} - \dfrac{y}{t^2} = \dfrac{1}{t^3}$

8. $\dfrac{dy}{dt} = y + e^{2t}$

9. $\dfrac{dy}{dt} + y = 2 \cos t$

10. $t^3\dfrac{dy}{dt} + (2 - 3t^2)y = t^3$

11. $\dfrac{dy}{dt} + \dfrac{y}{t \ln t} = t^2$

12. $2y \cot 2t + 1 - 2t \cot 2t - 2 \csc 2t = \dfrac{dy}{dt}$

In Exercises 13–15 solve the initial-value problem.

13. $\dfrac{dy}{dt} + 3t^2y = t^2, \quad y(1) = 2$

14. $(e^t \sin t - y) = \dfrac{dy}{dt}, \quad y(0) = -1$

15. $\dfrac{dy}{dt} + \dfrac{t^3y}{t^4 + 1} = t^7, \quad y(0) = 1$

16. When a substance such as glucose is administered intravenously into the bloodstream at a constant rate R, and the body uses the substance, the amount $A(t)$ in the blood satisfies the initial-value problem

$$\frac{dA}{dt} = R - kA, \qquad A(0) = A_0,$$

where $k > 0$ is a constant, and A_0 is the amount in the blood at time $t = 0$. Find $A(t)$.

17. Repeat Exercise 16 if R is a function of time.

18. A tank contains 1000 litres of water in which 5 kilograms of salt has been dissolved. A brine mixture containing 2 kilograms of salt for each 100 litres of solution is poured into the tank at 10 millilitres per second. At the same time, mixture is being drawn from the bottom of the tank at 5 millilitres per second. Assuming that the mitture in the tank is always well-stirred, the initial-value problem for the number of grams $S(t)$ of salt in the tank is

$$\frac{dS}{dt} + \frac{5S}{10^6 + 5t} = \frac{1}{5}, \qquad S(0) = 5000.$$

Find $S(t)$.

19. When a mass m falls under gravity, and it experiences a retarding force due to air resistance that is proportional to velocity, its velocity must satisfy the initial-value problem

$$m\frac{dv}{dt} = mg - kv, \qquad v(0) = v_0,$$

where $g = 9.81$, $k > 0$ is a constant, and v_0 is its velocity at time $t = 0$ when fall commences. Find $v(t)$.

Answers

1. $y = 2 + Ce^{-t^2}$ **2.** $y = t^4 + C/t^2$ **3.** $y = t/2 - 1/4 + Ce^{-2t}$ **4.** $y = \csc t(C - 5e^{\cos t})$
5. $y = (3C - t^3)/(3t^2 + 3)$ **6.** $y = -2(t + 1) + C(t + 1)^2$ **7.** $y = Ct - 1/(2t)$ **8.** $y = e^{2t} + Ce^t$
9. $y = \cos t + \sin t + Ce^{-t}$ **10.** $y = t^3/2 + Ct^3e^{1/t^2}$ **11.** $y = t^3/3 + (9C - t^3)/(9 \ln t)$
12. $y = t + \cos 2t + C \sin 2t$ **13.** $y = (1 + 5e^{1-t^3})/3$ **14.** $y = e^t(2 \sin t - \cos t)/5 - (4/5)e^{-t}$
15. $y = (t^4 + 1)^2/9 - (t^4 + 1)/5 + (49/45)(t^4 + 1)^{-1/4}$ **16.** $A = (R/k)(1 - e^{-kt}) + A_0e^{-kt}$
17. $A = \displaystyle\int_0^t R(u)e^{k(u-t)}du + A_0e^{-kt}$ **18.** $S = \dfrac{1}{50}(10^6 + 5t) - \dfrac{15 \times 10^9}{10^6 + 5t}$ grams
19. $v = \dfrac{mg}{k} + \left(v_0 - \dfrac{mg}{k}\right) e^{-kt/m}$ m/s

§4.2 Systems of Linear First-order Differential Equations

In this section, we illustrate how linear algebra can be used to decouple systems of coupled linear first-order differential equations. In Figure 4.1, the left container contains 1000 litres of water in which has been dissolved 50 kilograms of potassium; the right container contains 2000 litres of pure water. At time $t = 0$, a mixture containing 10 kilograms of potassium for each 100 litres of solution is pumped into the left container at the rate of 20 litres per minute. Well-stirred mixture from the left container enters the right container at 10 litres per minute and 5 litres per minute are pumped from right to left. Finally, mixture is removed from the left container at 15 litres per minute and 5 litres per minute from the right container. As a result, there is no increase in volumes in the containers; 25 litres enter the left container each minute and 25 litres leave; 10 litres enter the right container each minute and 10 litres leave. Suppose we let $p_1(t)$ and $p_2(t)$ be the amounts of potassium (in kilograms) in the tanks as functions of time (in minutes). These functions can be determined by using the fact that derivatives of $p_1(t)$ and $p_2(t)$ must be equal to the rate at which potassium enters each container less the rate at which it leaves.

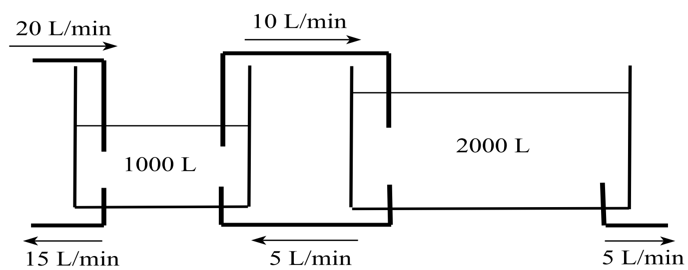


Figure 4.1

Potassium is pumped into the left container at $20(10/100) = 2$ kg/min. It also enters from the right container at $5(p_2/2000) = p_2/400$ kg/min. It is removed from the left container at $15(p_1/1000) = 3p_1/200$ kg/min and pumped into the right container at $10(p_1/1000) = p_1/100$ kg/min. Thus, for the left container,

$$\begin{aligned}\frac{dp_1}{dt} &= \begin{pmatrix}\text{rate at which}\\ \text{potassium enters}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{potassium leaves}\end{pmatrix}\\ &= \left(2 + \frac{p_2}{400}\right) - \left(\frac{3p_1}{200} + \frac{p_1}{100}\right)\\ &= -\frac{p_1}{40} + \frac{p_2}{400} + 2. \end{aligned} \tag{4.4a}$$

Potassium is pumped into the right container from the left at $p_1/100$ kg/min. It is pumped back into the left at $p_2/400$ kg/min and is drawn off for other purposes at the same rate. Thus,

$$\begin{aligned}\frac{dp_2}{dt} &= \begin{pmatrix}\text{rate at which}\\ \text{potassium enters}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{potassium leaves}\end{pmatrix}\\ &= \frac{p_1}{100} - \frac{p_2}{200}. \end{aligned} \tag{4.5b}$$

When the initial conditions

$$p_1(0) = 50, \qquad p_2(0) = 0, \tag{4.5c}$$

are added, we have a system of two coupled, first-order, linear, differential equations in $p_1(t)$ and $p_2(t)$. In a differential equations course, these would be solved using operators, Laplace transforms, and/or matrices. In the operator approach, the equations are decoupled but the result is a second order equation in either $p_1(t)$ or $p_2(t)$. Laplace transforms do not decouple the equations; the transform replaces the system of differential equations with a system of algebraic equations in the transforms of $p_1(t)$ and $p_2(t)$. In the matrix method, the system is written in the form

$$\frac{d\mathbf{p}}{dt} = A\mathbf{p} + \mathbf{b} \quad\Longleftrightarrow\quad \frac{d}{dt}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -1/40 & 1/400 \\ 1/100 & -1/200 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \tag{4.6}$$

Eigenvalues and eigenvectors of the coefficient matrix A yield solutions of this vector differential equation, and components then give solutions of the scalar equations. We use eigenpairs in a fundamentally different way here to decouple the equations into first-order equations. We will return to this example later when we have illustrated the technique in examples with less cumbersome calculations.

Consider the system of coupled, first-order differential equations

$$\frac{dy_1}{dt} = y_1 + 4y_2 + 2, \qquad \frac{dy_2}{dt} = 2y_1 + 3y_2 - t,$$

for $y_1(t)$ and $y_2(t)$, subject to the initial conditions $y_1(0) = 0$ and $y_2(0) = 1$. We can write the system in matrix form as

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} + \mathbf{b} \quad\Longleftrightarrow\quad \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -t \end{pmatrix}. \tag{4.7}$$

Eigenvalues of A are defined by

$$0 = \det\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 8 = (\lambda+1)(\lambda-5).$$

Eigenvectors corresponding to $\lambda_1 = -1$ are multiples of $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, and those corresponding to $\lambda_2 = 5$ are multiples of $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Let $q_1(t)$ and $q_2(t)$ denote the components of $\mathbf{y}(t)$ with respect to the basis of eigenvectors,

$$\mathbf{y}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2.$$

If b_1 and b_2 are the components of the vector $\mathbf{b} = (2, -t)$, with respect to the eigenvector basis, then

$$\begin{pmatrix} 2 \\ -t \end{pmatrix} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 = b_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} + b_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Equating coefficients leads to $b_1 = -(t+2)/3$ and $b_2 = (2-2t)/3$. We could also have used the transition matrix

$$T = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}$$

to find the components of $\mathbf{b}$ with respect to the eigenvector basis,

$$T\mathbf{b} = -\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ -t \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 2+t \\ -2+2t \end{pmatrix}.$$

Equation 4.7 is a vector differential equation for the components $y_1(t)$ and $y_2(t)$ of $\mathbf{y}(t)$ with respect to the basis $\{(1,0),(0,1)\}$. When we use the eigenvectors as a basis, the differential equation becomes

$$\frac{d}{dt}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 2+t \\ -2+2t \end{pmatrix}.$$

Equating entries gives the system of differential equations

$$\frac{dq_1}{dt} = -q_1 - \frac{1}{3}(2+t), \qquad \frac{dq_2}{dt} = 5q_2 + \frac{2}{3}(1-t).$$

The eigenvalue basis has decoupled the system into linear first-order differential equations in $q_1(t)$ and $q_2(t)$. When integrating factors are calculated, solutions are

$$q_1(t) = C_1e^{-t} - \frac{1}{3}(t+1), \qquad q_2(t) = C_2e^{5t} + \frac{2}{75}(5t-4),$$

where C_1 and C_2 are constants. To evaluate C_1 and C_2, we can transform the initial conditions to the eigenvector basis, or return to the natural basis. To do the latter we write

$$\mathbf{y}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2 = \left[C_1e^{-t} - \frac{1}{3}(t+1)\right]\begin{pmatrix}-2\\1\end{pmatrix} + \left[C_2e^{5t} + \frac{2}{75}(5t-4)\right]\begin{pmatrix}1\\1\end{pmatrix}.$$

When we take components, we obtain

$$y_1(t) = -2C_1e^{-t} + \frac{2}{3}(t+1) + C_2e^{5t} + \frac{2}{75}(5t-4) = -2C_1e^{-t} + C_2e^{5t} + \frac{2}{25}(10t+7),$$
$$y_2(t) = C_1e^{-t} - \frac{1}{3}(t+1) + C_2e^{5t} + \frac{2}{75}(5t-4) = C_1e^{-t} + C_2e^{5t} - \frac{1}{25}(5t+11).$$

The initial conditions require

$$0 = y_1(0) = -2C_1 + C_2 + \frac{14}{25}, \qquad 1 = y_2(0) = C_1 + C_2 - \frac{11}{25}.$$

These give $C_1 = 1/3$ and $C_2 = 8/75$. The solution of the system of differential equations is

$$y_1(t) = -\frac{2}{3}e^{-t} + \frac{8}{75}e^{5t} + \frac{2}{25}(10t+7), \quad y_2(t) = \frac{1}{3}e^{-t} + \frac{8}{75}e^{5t} - \frac{1}{25}(5t+11).$$

We are now ready to return to the containers with potassium. Eigenvalues of matrix A in equation 4.6 are defined by

$$0 = \det\begin{pmatrix}-1/40-\lambda & 1/400\\ 1/100 & -1/200-\lambda\end{pmatrix}.$$

They are $\lambda = \dfrac{-3\pm\sqrt{5}}{200}$. We denote them by $\lambda_1 = \dfrac{-3+\sqrt{5}}{200}$ and $\lambda_2 = -\dfrac{3+\sqrt{5}}{200}$. Corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix}1\\4+2\sqrt{5}\end{pmatrix}$, and $\mathbf{v}_2 = \begin{pmatrix}1\\4-2\sqrt{5}\end{pmatrix}$. Let $q_1(t)$ and $q_2(t)$ denote the components of $\mathbf{p}(t)$ with respect to the basis of eigenvectors,

$$\mathbf{p}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2.$$

If b_1 and b_2 are the components of the vector $\mathbf{b} = (2,0)$, with respect to the eigenvector basis, then

$$\begin{pmatrix}2\\0\end{pmatrix} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 = b_1\begin{pmatrix}1\\4+2\sqrt{5}\end{pmatrix} + b_2\begin{pmatrix}1\\4-2\sqrt{5}\end{pmatrix}.$$

Equating coefficients leads to $b_1 = 1 - 2/\sqrt{5}$ and $b_2 = 1 + 2/\sqrt{5}$. (Once again we could have found the transition matrix from the natural basis to the eigenvector basis.) Equation 4.6 is a vector differential equation for the components $p_1(t)$ and $p_2(t)$ of $\mathbf{p}(t)$ with respect to the basis $\{(1,0),(0,1)\}$. When we use the eigenvectors as a basis, the differential equation becomes

$$\frac{d}{dt}\begin{pmatrix}q_1\\q_2\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix} + \begin{pmatrix}b_1\\b_2\end{pmatrix}.$$

Equating entries gives the system of differential equations

$$\frac{dq_1}{dt} = \lambda_1 q_1 + b_1, \qquad \frac{dq_2}{dt} = \lambda_2 q_2 + b_2.$$

The eigenvalue basis has decoupled the system into linear first-order differential equations in $q_1(t)$ and $q_2(t)$. When integrating factors are calculated, solutions are

$$q_1(t) = C_1 e^{\lambda_1 t} - \frac{b_1}{\lambda_1}, \qquad q_2(t) = C_2 e^{\lambda_2 t} - \frac{b_2}{\lambda_2},$$

where C_1 and C_2 are constants. To evaluate C_1 and C_2, we can transform the initial conditions to the eigenvector basis, or return to the natural basis. To use the natural basis, we write

$$\mathbf{p}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2 = \left(C_1 e^{\lambda_1 t} - \frac{b_1}{\lambda_1}\right)\begin{pmatrix} 1 \\ 4+2\sqrt{5} \end{pmatrix} + \left(C_2 e^{\lambda_2 t} - \frac{b_2}{\lambda_2}\right)\begin{pmatrix} 1 \\ 4-2\sqrt{5} \end{pmatrix}.$$

The initial conditions require

$$\begin{pmatrix} 50 \\ 0 \end{pmatrix} = \left(C_1 - \frac{b_1}{\lambda_1}\right)\begin{pmatrix} 1 \\ 4+2\sqrt{5} \end{pmatrix} + \left(C_2 - \frac{b_2}{\lambda_2}\right)\begin{pmatrix} 1 \\ 4-2\sqrt{5} \end{pmatrix}.$$

When we equate components and solve for C_1 and C_2, we get $C_1 = -25$ and $C_2 = -25$. With $b_1/\lambda_1 = 10(\sqrt{5} - 5)$ and $b_2/\lambda_2 = -10(\sqrt{5} + 5)$, we obtain

$$\mathbf{p}(t) = \left[-25e^{\lambda_1 t} - 10(\sqrt{5} - 5)\right]\begin{pmatrix} 1 \\ 4+2\sqrt{5} \end{pmatrix} + \left[-25e^{\lambda_2 t} + 10(\sqrt{5} + 5)\right]\begin{pmatrix} 1 \\ 4-2\sqrt{5} \end{pmatrix}.$$

The numbers of kilograms of potassium in the tanks is therefore

$$\begin{aligned} p_1(t) &= -25e^{\lambda_1 t} + 10(5 - \sqrt{5}) - 25e^{\lambda_2 t} + 10(\sqrt{5} + 5) \\ &= -25e^{\lambda_1 t} - 25e^{\lambda_2 t} + 100, \\ p_2(t) &= (4 + 2\sqrt{5})[-25e^{\lambda_1 t} + 10(5 - \sqrt{5}] + (4 - 2\sqrt{5})[-25e^{\lambda_2 t} + 10(\sqrt{5} + 5)] \\ &= -50(2 + \sqrt{5})e^{\lambda_1 t} - 50(2 - \sqrt{5})e^{\lambda_2 t} + 200. \end{aligned}$$

Since λ_1 and λ_2 are both negative, limits of $p_1(t)$ and $p_2(t)$ for large t are 100 and 200 kilograms respectively. These functions are plotted in Figure 4.2.

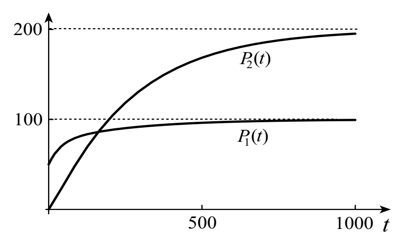


Figure 4.2

EXERCISES 4.2

In Exercises 1–4 use eigenvectors to find the solution of the initial-value problem.

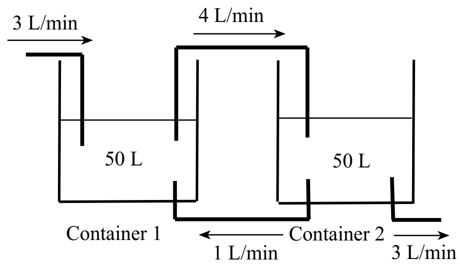
1. $\dfrac{dy_1}{dt} = y_1 + y_2, \quad \dfrac{dy_2}{dt} = 4y_1 - 2y_2$; $\quad y_1(0) = 1, \quad y_2(0) = 6$

2. $\dfrac{dy_1}{dt} = y_1 + 3y_2 + 2t + 3, \quad \dfrac{dy_2}{dt} = 4y_1 + 5y_2 - t$; $\quad y_1(0) = -1, \quad y_2(0) = 1$

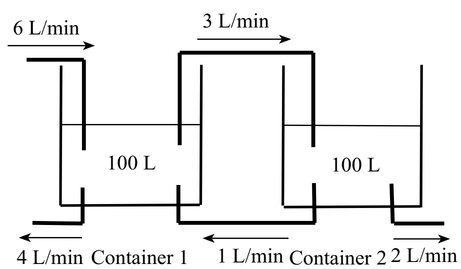
3. $\dfrac{dy_1}{dt} = 4y_1 + 2y_2 + 2y_3, \quad \dfrac{dy_2}{dt} = 2y_1 + 4y_2 + 2y_3, \quad \dfrac{dy_3}{dt} = 2y_1 + 2y_2 + 4y_3$; $\quad y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 1$

4. $\dfrac{dy_1}{dt} = 4y_1 + y_3 + 1, \quad \dfrac{dy_2}{dt} = -2y_1 + y_2 - 3, \quad \dfrac{dy_3}{dt} = -2y_1 + y_3$; $\quad y_1(0) = -1, \quad y_2(0) = 1, \quad y_3(0) = 0$

5. Container 1 in the figure below contains 50 litres of water with 25 kilograms of dissolved sugar. Container 2 has 50 litres of pure water. Pure water is added to container 1 at 3 litres per minute. Well-stirred mixture is pumped from container 1 into container 2 at 4 litres per minute, and simultaneously from container 2 into container 1 at 1 litre per minute. Mixture is also drawn from container 2 at 3 litres per minute. Find the numbers of kilograms of sugar in the containers as functions of time t. Are there limits as $t \to \infty$?

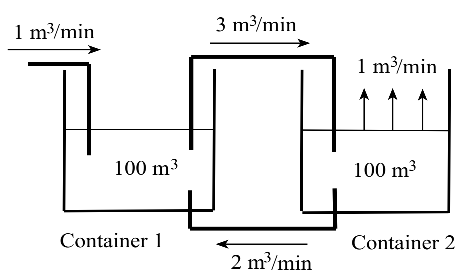


6. Container 1 in the figure below contains 100 litres of pure water; container 2 has 100 litres of water in which has been dissolved 200 kilograms of salt. Solution with 2 kilograms of salt per litre of solution is pumped into container 1 at 6 litres per minute. Well-stirred mixture is pumped from container 1 into container 2 at 3 litres per minute, and simultaneously from container 2 into container 1 at 1 litre per minute. Mixtures are drawn from containers 1 and 2 at 4 and 2 litres per minute respectively. Find the numbers of kilograms of salt in the containers as functions of time t. Are there limits as $t \to \infty$?

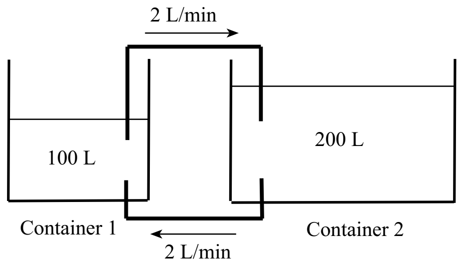


7. Repeat Exercise 5 if pure water entering container 1 is replaced by a solution containing 2 kilograms of sugar per litre of liquid.

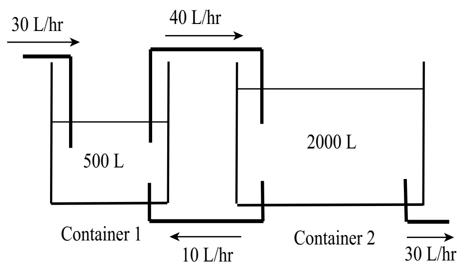
8. The containers below each have 100 cubic metres of pure water. A solution containing 1 kilogram of salt per cubic metre of liquid is added to container 1 at a rate of 1 cubic metre per minute. Well-stirred mixture is pumped from container 1 to container 2 at 3 cubic metres per minute. Water evaporates from container 2 at 1 cubic metre per minute, and mixture is pumped from container 2 to container 1 at 2 cubic metre per minute. Find the numbers of kilograms of salt in the containers as functions of time t. Are there limits as $t \to \infty$?



9. Container 1 in the figure below contains 100 litres of solution with salt concentration 1/2 kilogram per litre. Container 2 has 200 litres of solution with concentration 1/10 kilogram per litre. Solution is pumped from container 1 to container 2 at 2 litres per minute, and back to container 1 at the same rate. Find the numbers of kilograms of salt in the containers as functions of time t. Are there limits as $t \to \infty$?



10. Container 1 below has 500 kilograms of potassium dissolved in 500 litres of water. Container 2 has 2000 litres of pure water. Pure water is added to container 1 at 30 litres per hour, and the well-stirred mixture is pumped into container 2 at 40 litres per hour. Solution is pumped back from container 2 to container 1 at 10 litres per hour, and also removed at 30 litres per hour. Find the maximum amount of potassium ever found in container 2.



11. One tank contains 150 cubic metres of pure water and a second contains 150 cubic metres of water in which 100 kilograms of salt has been dissolved. Well-stirred mixtures circulate from each tank to the other at the rate of 10 cubic metres per hour. When does the tank that started with no salt have 25 kilograms of salt?

12. You must have noticed that in all mixing problems, the volumes in containers never changes. Our decoupling method is not applicable to problems with changing volumes. To illustrate, suppose the situation in Figure 4.1 is the same except that mixture is not returned from the right container to the left container, nor drawn from either container, until the volume of solution in the right container reaches 2500 litres.
(a) Set up a system of differential equations for the amount of potassium in each container.
(b) Explain why neither operators nor Laplace transforms are conducive to the system.

Answers

1. $y_1(t) = 2e^{2t} - e^{-3t}$, $y_2(t) = 2e^{2t} + 4e^{-3t}$
2. $y_1(t) = -(3/2)e^{-t} + (23/98)e^{7t} + 13(7t+1)/49$, $y_2(t) = e^{-t} + (23/49)e^{7t} - (63t+23)/49$
3. $y_1(t) = (2/3)e^{8t} + (1/3)e^{2t}$, $y_2(t) = (2/3)e^{8t} - (2/3)e^{2t}$, $y_3(t) = (2/3)e^{8t} + (1/3)e^{2t}$
4. $y_1(t) = -(4/3)e^{3t} + (1/2)e^{2t} - 1/6$, $y_2(t) = (4/3)e^{3t} - e^{2t} - 2e^{t} + 8/3$, $y_3(t) = (4/3)e^{3t} - e^{2t} - 1/3$
5. $(25/2)(e^{-t/25} + e^{-3t/25})$, $25(e^{-t/25} - e^{-3t/25})$; 0, 0
6. $200 + (200/\sqrt{7} - 100)e^{(-5+\sqrt{7})t/100} - (200/\sqrt{7} + 100)e^{-(5+\sqrt{7})t/100}$
$200 - (300/\sqrt{7})e^{(-5+\sqrt{7})t/100} + (300/\sqrt{7})e^{-(5+\sqrt{7})t/100}$; 200, 200
7. $100 - (125/2)e^{-t/25} - (25/2)e^{-3t/25}$, $100 + 25e^{-3t/25} - 125e^{-t/25}$; 100, 100
8. $12 - 12e^{-t/20} + 2t/5$, $-12 + 12e^{-t/20} + 3t/5$; no limits
9. $70/3 + (80/3)e^{-3t/100}$, $140/3 - (80/3)e^{-3t/100}$; 70/3, 140/3
10. $[1000(5+\sqrt{13})/3][(5-\sqrt{13})/(5+\sqrt{13})]^{(5+\sqrt{13})/(2\sqrt{13})}$ kg
11. $(15/2)\ln 2$ hours
12.(a) $\dfrac{dP_1}{dt} = 2 - \dfrac{P_1}{100+t} + 2 + \dfrac{P_2}{500} - \dfrac{P_1}{60}$, $\quad \dfrac{dP_2}{dt} = \dfrac{P_1}{100+t} + \dfrac{P_1}{150} - \dfrac{P_2}{250}$
(b) When we rewrite the system in matrix form, the matrix of coefficients is not time independent.

§4.3 Linear Second-order Differential Equations

Linear second-order differential equations are of the form

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = F(t), \tag{4.8}$$

where $P(t)$, $Q(t)$, $R(t)$, and $F(t)$ are given functions. The equation is said to be homogeneous if $F(t) \equiv 0$, and nonhomogeneous otherwise. Even the homogeneous equation

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0, \tag{4.9}$$

which is simpler than the corresponding nonhomogeneous equation, can be notoriously difficult to solve, but it can also be very easy to solve. The fact that the equation is linear leads to the *superposition principle*.

Theorem 4.1 (**Superposition Principle**) If $y_1(t)$ and $y_2(t)$ are any two solutions of differential equation 4.9, then so also is any linear combination of them, namely, $c_1y_1(t) + c_2y_2(t)$ for any constants c_1 and c_2. Furthermore, any such linear combination of two linearly independent solutions is a general solution of the differential equation.

What this means is that to find a general solution of differential equation 4.9, we need only find two linearly independent solutions $y_1(t)$ and $y_2(t)$; their superposition gives a general solution.

Closure of the set of solutions under linear combinations suggests, and it is straightforward to check, that the set of solutions of equation 4.9 is a real vector space with solutions being vectors in the space. Any two linearly independent solutions constitute a basis for the space, which therefore has dimension two.

One case when equation 4.9 is easy to solve is when P, Q, and R are constant functions. This places severe restriction on P, Q and R, but the resulting equation has many important applications. Consider then the homogeneous equation

$$p\frac{d^2y}{dt^2} + q\frac{dy}{dt} + ry = 0, \tag{4.10}$$

where p, q, and r are constants. To find linearly independent solutions $y_1(t)$ and $y_2(t)$ of this equation, we look for solutions of the form $y(t) = e^{mt}$, for some constant m. When we substitute this into the differential equation, we obtain

$$0 = pm^2e^{mt} + qme^{mt} + re^{mt} = e^{mt}(pm^2 + qm + r).$$

Since the exponential cannot vanish, we must set $pm^2 + qm + r = 0$. This is called the **auxiliary equation** associated with the differential equation. If m is a real root of the auxiliary equation, then e^{mt} is a solution of the differential equation. If the auxiliary equation has two real, distinct roots, then a general solution of the differential equation results. But the reader must realize that the quadratic equation could have equal real roots, and/or complex roots, and in such cases a general solution does not immediately result. We will be able to adapt to these situations and still find a general solution of the differential equation. We consider some examples before stating general results. The auxiliary equation associated with the differential equation

$$y'' - 2y' - 3y = 0$$

is $0 = m^2 - 2m - 3 = (m-3)(m+1)$ with roots $m = -1$ and $m = 3$. Thus, $y_1(t) = e^{-t}$ and $y_2(t) = e^{3t}$ are linearly independent solutions of the differential equation, and a general solution is $y(t) = c_1e^{-t} + c_2e^{3t}$.

The auxiliary equation associated with the differential equation

$$y'' + 6y' + 9y = 0$$

is $0 = m^2 + 6m + 9 = (m+3)^2$ with equal roots $m = -3, -3$. Only one solution of the form e^{mt} is found, $y_1(t) = e^{-3t}$. We need a second solution to superpose with $y_1(t)$ to obtain a general solution. Suppose we are sufficiently intuitive to think that a second solution might be of the form $y_2(t) = u(t)e^{-3t}$; that is, the product of some yet to be determined function $u(t)$ times the already known solution $y_1(t)$. If we substitute this into the differential equation, we find

$$0 = [u''e^{-3t} - 6u'e^{-3t} + 9ue^{-3t}] + 6[u'e^{-3t} - 3ue^{-3t}] + 9ue^{-3t} = u''e^{-3t}.$$

This implies that $u''(t) = 0$, from which $u(t) = c_1 t + c_2$, where c_1 and c_2 are constants. If we set $c_1 = 0$ and $c_2 = 1$, then $y_2(t) = e^{-3t}$, our original solution $y_1(t)$. If we set $c_1 = 1$ and $c_2 = 0$, we obtain a new solution $y_2(t) = te^{-3t}$, a solution independent of $y_1(t)$. In other words, a pair of linearly independent solutions of the differential equation is $y_1(t) = e^{-3t}$ and $y_2(t) = te^{-3t}$. A general solution is their superposition, $y(t) = c_1 e^{-3t} + c_2 te^{-3t} = (c_1 + c_2 t)e^{-3t}$.

For our final example, consider

$$y'' + 2y' + 10y = 0.$$

The auxiliary equation is $0 = m^2 + 2m + 10$ with roots

$$m = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm 3i.$$

This means that there is no real exponential $y = e^{mt}$ that satisfies the differential equation. If, however, we form complex exponentials $y_1(t) = e^{(-1+3i)t}$ and $y_2(t) = e^{(-1-3i)t}$, and superpose these solutions, then

$$y(t) = Ae^{(-1+3i)t} + Be^{(-1-3i)t}$$

must also be a solution for any pair of constants A and B, real or complex. In general, this solution is a complex one, and we mean complex, as opposed to real, not complex because it is difficult. But we are interested in real solutions of the (real) differential equation, not complex ones. We now show how to derive real solutions from this complex one using Euler's identity, $e^{\theta i} = \cos\theta + i\,\sin\theta$. If we write the complex solution in the form

$$y(t) = Ae^{-t}e^{3ti} + Be^{-t}e^{-3ti},$$

and use Euler's identity on e^{3ti} and e^{-3ti}, we get

$$\begin{aligned} y(t) &= Ae^{-t}(\cos 3t + i\,\sin 3t) + Be^{-t}(\cos(-3t) + i\,\sin(-3t)) \\ &= Ae^{-t}(\cos 3t + i\,\sin 3t) + Be^{-t}(\cos 3t - i\,\sin 3t) \\ &= e^{-t}[(A+B)\,\cos 3t + i(A-B)\,\sin 3t]. \end{aligned}$$

Suppose now that c_1 and c_2 are any real constants whatsoever. If we set $A = (c_1 - c_2 i)/2$ and $B = (c_1 + c_2 i)/2$, then $A + B = c_1$ and $i(A - B) = c_2$. Thus, the function

$$y(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

is a solution of the differential equation for any real constants c_1 and c_2; that is, it is real, and it is a general solution of the differential equation. It has been derived from the complex roots $m = -1 \pm 3i$ of the equation $m^2 + 2m + 10 = 0$. Note that what multiplies t in the exponential is the real part of these complex numbers, and what multiplies t in the trigonometric functions is the imaginary part.

What we have seen in these three examples is typical for all linear, second-order, homogeneous, differential equations 4.10. We summarize results in the following theorem.

Theorem 4.2 If $pm^2 + qm + r = 0$ is the auxiliary equation associated with differential equation 4.10, then:

1. when roots m_1 and m_2 are real and distinct, a general solution of the differential equation is

$$y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}; \tag{4.11a}$$

2. when m is a double root, a general solution of the differential equation is

$$y(t) = (c_1 + c_2 t)e^{mt}; \tag{4.12b}$$

3. when roots are complex $a \pm bi$, a general solution of the differential equation is

$$y(t) = e^{at}(c_1 \cos bt + c_2 \sin bt). \tag{4.13c}$$

Here are some further examples.

Example 4.3 Find general solutions for each of the following differential equations:

(a) $2y'' + 4y' - 7y = 0$ (b) $y'' + 10y' + 25y = 0$ (c) $3y'' + 3y' + 5y = 0$

Solution (a) The auxiliary equation is $2m^2 + 4m - 7 = 0$ with solutions

$$m = \frac{-4 \pm \sqrt{16 + 56}}{4} = \frac{-2 \pm 3\sqrt{2}}{2}.$$

A general solution of the differential equation is $y(t) = c_1 e^{(-2+3\sqrt{2})t/2} + c_2 e^{-(2+3\sqrt{2})t/2}$.
(b) The auxiliary equation is $0 = m^2 + 10m + 25 = (m + 5)^2$ with solutions $m = -5, -5$. A general solution of the differential equation is $y(t) = (c_1 + c_2 t)e^{-5t}$.
(c) The auxiliary equation is $3m^2 + 3m + 5 = 0$ with solutions

$$m = \frac{-3 \pm \sqrt{9 - 60}}{6} = \frac{-3 \pm \sqrt{51}i}{6}.$$

A general solution of the differential equation is $y(t) = e^{-t/2}\left(c_1 \cos \frac{\sqrt{51}t}{6} + c_2 \sin \frac{\sqrt{51}t}{6}\right)$.●

Nonhomogeneous Linear Differential Equations

We now consider the situation when differential equation 4.8 is nonhomogeneous, so that $F(t)$ is not identically zero. The following theorem tells us how to solve such equations.

Theorem 4.3 A general solution of the differential equation 4.8 is $y(t) = y_h(t) + y_p(t)$, where $y_h(t)$ is a general solution of the associated homogeneous equation (the equation obtained when $F(t)$ is replaced by 0), and $y_p(t)$ is any particular solution of the given equation.

Because of this theorem discussions of nonhomogeneous differential equations can be divided into two parts. First, find a general solution $y_h(t)$ of the associated homogeneous equation. To this, add any particular solution $y_p(t)$ of nonhomogeneous equation 4.8. There are a number of methods for finding $y_p(t)$, including undetermined coefficients, annihilators, operators, and variation of parameters. Each method has its advantages and disadvantages. We discuss only the method of undetermined coefficients. It is applicable to differential equations with constant coefficients but, in addition, $F(t)$ must be a power t^n, n a nonnegative integer, an exponential e^{pt}, a sine $\sin pt$, a cosine $\cos pt$, and/or any sums or products thereof.

Method of Undetermined Coefficients for a Particular Solution

We illustrate the method before stating two general rules. For the equation

$$y'' + y' - 6y = e^{4t},$$

the method essentially says that Ae^{4t} is the simplest function that could conceivably yield e^{4t} when substituted into the left side of the differential equation. Consequently, it is natural to assume that $y_p = Ae^{4t}$ and attempt to determine the unknown coefficient A. Substitution of this function into the differential equation gives

$$16Ae^{4t} + 4Ae^{4t} - 6Ae^{4t} = e^{4t}.$$

If we divide by e^{4t}, then

$$14A = 1 \quad \text{and} \quad A = \tfrac{1}{14}.$$

A particular solution is therefore $y_p = e^{4t}/14$.

We illustrate a few more possibilities in the following example.

Example 4.4 Find a particular solution of $y'' + y' - 6y = F(t)$ when:

(a) $F(t) = 6t^2 + 2t + 3$ (b) $F(t) = 2\sin 2t$ (c) $F(t) = te^{-t} - e^{-t}$

Solution (a) Since terms in t^2, t, and constants yield terms in t^2, t, and constants when substituted into the left side of the differential equation, we attempt to find a particular solution of the form $y_p = At^2 + Bt + C$. Substitution into the differential equation gives

$$(2A) + (2At + B) - 6(At^2 + Bt + C) = 6t^2 + 2t + 3,$$

or

$$(-6A)t^2 + (2A - 6B)t + (2A + B - 6C) = 6t^2 + 2t + 3.$$

Since the functions $\{1, t, t^2\}$ are linearly independent, this equation can hold for all values of t only if coefficients of corresponding powers of t are identical. Equating coefficients then gives

$$-6A = 6, \qquad 2A - 6B = 2, \qquad 2A + B - 6C = 3.$$

These imply that $A = -1, B = -2/3, C = -17/18$, and

$$y_p = -t^2 - \frac{2t}{3} - \frac{17}{18}.$$

(b) Since terms in $\sin 2t$ and $\cos 2t$ yield terms in $\sin 2t$ when substituted into the left side of the differential equation, we assume that $y_p = A\sin 2t + B\cos 2t$. Substitution into the differential equation gives

$$(-4A\sin 2t - 4B\cos 2t) + (2A\cos 2t - 2B\sin 2t) \\ - 6(A\sin 2t + B\cos 2r) = 2\sin 2t,$$

or

$$(-10A - 2B)\sin 2t + (2A - 10B)\cos 2t = 2\sin 2t.$$

Equating coefficients of $\sin 2t$ and $\cos 2t$ gives

$$-10A - 2B = 2, \qquad 2A - 10B = 0.$$

These imply that $A = -5/26, B = -1/26$, and hence

$$y_p = -\frac{1}{26}(5\sin 2t + \cos 2t).$$

(c) Since terms in te^{-t} and e^{-t} yield terms in te^{-t} and e^{-t} when substituted into the left side of the differential equation, we assume that $y_p = Ate^{-t} + Be^{-t}$. Substitution into the differential equation gives

$$(Ate^{-t} - 2Ae^{-t} + Be^{-t}) + (-Ate^{-t} + Ae^{-t} - Be^{-t})$$
$$- 6(Ate^{-t} + Be^{-t}) = te^{-t} - e^{-t},$$

or

$$(-6A)te^{-t} + (-A - 6B)e^{-t} = te^{-t} - e^{-t}.$$

Equating coefficients of e^{-t} and te^{-t} yields

$$-6A = 1, \qquad -A - 6B = -1.$$

These imply that $A = -1/6, B = 7/36$, and hence

$$y_p = -(1/6)te^{-t} + (7/36)e^{-t}.\bullet$$

The following rule encompasses each part of this example.

Rule 1

If a term of $F(t)$ consists of a power (t^n), an exponential (e^{pt}), a sine ($\sin pt$), a cosine ($\cos pt$), or any product thereof, assume as a part of y_p a constant multiplied by that term plus a constant multiplied by any linearly independent function arising from it by differentiation.

For Example 4.4(a), since $F(t)$ contains the term $6t^2$, we assume y_p contains At^2. Differentiation of At^2 yields a term in t and a constant so that we form $y_p = At^2+Bt+C$. No new terms for y_p are obtained from the terms $2t$ and 3 in $F(t)$.

For Example 4.4(b), we assume that y_p contains $A\sin 2t$ to account for the term $2\sin 2t$ in $F(t)$. Differentiation of $A\sin 2t$ gives a linearly independent term in $\cos 2t$ so that we form $y_p = A\sin 2t + B\cos 2t$.

For Example 4.4(c), since $F(t)$ contains the term te^{-t}, we assume that y_p contains Ate^{-t}. Differentiation of Ate^{-t} yields a term in e^{-t} so that we form $y_p = Ate^{-t} + Be^{-t}$. No new terms for y_p are obtained from the term $-e^{-t}$ in $F(t)$.

Example 4.5 What is the form of the particular solution predicted by Rule 1 for the differential equation

$$y'' + 15y' - 6y = t^2e^{4t} + t + t\cos t?$$

Solution Rule 1 suggests that

$$y_p = At^2e^{4t} + Bte^{4t} + Ce^{4t} + Dt + E + Ft\cos t + Gt\sin t$$
$$+ H\cos t + I\sin t.\bullet$$

Unfortunately, exceptions to Rule 1 do occur. For the differential equation $y'' + y = \cos t$, Rule 1 would predict $y_p = A\cos t + B\sin t$. If we substitute this into the differential equation we obtain the absurd identity $0 = \cos t$, and certainly no equations to solve for A and B. This result could have been predicted had we first calculated $y_h(t)$. The auxiliary equation $m^2 + 1 = 0$ has solutions $m = \pm i$ so that $y_h(t) = C_1\cos t + C_2\sin t$. Since y_p as suggested by Rule 1 is precisely y_h with different names for the constants, then certainly $y_p'' + y_p = 0$. Suppose that as an alternative we multiply this y_p by t, and assume that $y_p = At\cos t + Bt\sin t$. Substitution into the differential equation now gives

$$-2A\sin t + 2B\cos t = \cos t.$$

Identification of coefficients requires $A = 0, B = 1/2$, and $y_p = (1/2)t\sin t$.

This example suggests that if y_p predicted by Rule 1 is already contained in y_h, then a modification of y_p is necessary. A precise statement of the situation is given in the following rule.

Rule 2

Suppose that a term in $F(t)$ is of the form $t^n f(t)$ (n a nonnegative integer). Begin by writing the form of a particular solution corresponding to this term as predicted by Rule 1. Now ask whether $f(t)$ can be obtained from $y_h(t)$ by specifying values for the arbitrary constants. If the answer is no, we have the correct form for the particular solution. If the answer is yes, we must modify the form of the particular solution by multiplying it by t^k where k is the multiplicity of the root of the auxiliary equation giving rise to $f(t)$.

Rule 2 always takes precedence over Rule 1. To use this rule we first require $y_h(t)$. Then, and only then, can we decide on the form of $y_p(t)$. As an illustration, consider the following example.

Example 4.6 Find a general solution for $y'' - 2y' + y = t^3 e^t$.

Solution The auxiliary equation is $m^2 - 2m + 1 = 0$ with root $m = 1$ of multiplicity 2, from which

$$y_h(t) = (C_1 + C_2 t)e^t.$$

According to Rule 1, the particular solution should be $At^3e^t + Bt^2e^t + Cte^t + De^t$. Because e^t can be obtained from y_h by specifying $C_1 = 1$ and $C_2 = 0$, Rule 2 must be invoked. Since this term results from the root $m = 1$ of multiplicity 2, we assume that

$$y_p = t^2(At^3e^t + Bt^2e^t + Cte^t + De^t) = At^5e^t + Bt^4e^t + Ct^3e^t + Dt^2e^t.$$

Substitution into the differential equation and simplification gives

$$20At^3e^t + 12Bt^2e^t + 6Cte^t + 2De^t = t^3e^t.$$

Equating coefficients gives

$$20A = 1, \quad 12B = 0, \quad 6C = 0, \quad 2D = 0.$$

Thus, $y_p(t) = (1/20)t^5e^t$, and a general solution of the differential equation is therefore

$$y(t) = (C_1 + C_2 t)e^t + \frac{1}{20}t^5e^t. \bullet$$

Applications of Linear Second Order Differential Equations

Many vibration problems give rise to linear, second-order, constant-coefficient differential equations, and such problems are often modeled, at least in the first instance, by vibrating mass-spring systems. Consider the situation in Figure 4.3 of a spring attached to a solid wall on one end and a mass on the other. If the mass is somehow set into horizontal motion along the axis of the spring it will continue to do so for some time. The nature of the motion depends on a number of factors such as the tightness of the spring, the amount of mass, whether there is friction between the mass and the surface along which it slides, whether there is friction between the mass and the medium in which it slides, and whether there are any other forces acting on the mass.

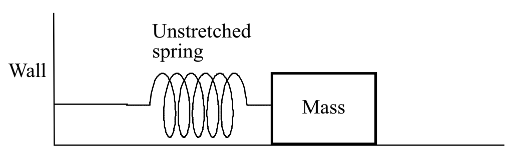


Figure 4.3

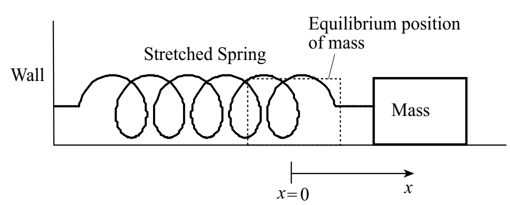


Figure 4.4

Our objective is to predict the position of the mass at any given time, knowing the forces acting on the mass, and how motion is initiated. We begin by establishing a means by which to identify the position of the mass. Most convenient is to let x represent the position of the mass relative to the position that it would occupy were the spring unstretched and uncompressed, called the **equilibrium position** (Figure 4.4). We shall then look for x as a function of time t, taking $t = 0$ at the instant that motion is initiated. To determine the differential equation describing oscillations of the mass, we analyze the forces acting on the mass when it is at position x. First there is the spring. Hooke's Law states that when a spring is stretched, the force exerted by the spring in an attempt to restore itself to an unstretched position is proportional to the amount of stretch in the spring. Since x not only identifies the position of the mass, but also represents the stretch in the spring, it follows that the restoring force exerted by the spring on the mass at position x is $-kx$, where $k > 0$ is the constant of proportionality, called the **spring constant**. The negative sign indicates that the force is to the left when x is positive and the spring is stretched. In a compressed situation, the spring force should be positive (to the right). This is indeed the case, because with compression, x is negative, and therefore $-kx$ is positive.

In many vibration problems, there is a **damping force**, a force opposing motion that has magnitude directly proportional to the velocity of the mass. It might be a result of air friction with the mass, or it might be due to a mechanical device like a shock absorber on a car, or a combination of such forces. Damping forces are modelled by what is called a **dashpot**; it is shown in (Figure 4.5). Because damping forces are proportional to velocity, and velocity is given by dx/dt, they can be represented in the form $-\beta(dx/dt)$, where $\beta > 0$ is a constant. The negative sign accounts for the fact that damping forces oppose motion; they are in the opposite direction to velocity.

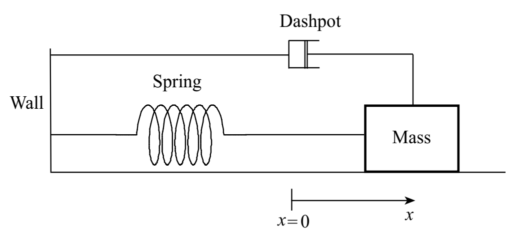


Figure 4.5

There could be other forces that act on the mass; they could depend on both the position of the mass and time. In the event that they depend only on time, we denote them by $F(t)$. The total force acting on the mass is therefore $-kx - \beta(dx/dt) + F(t)$. According to Newton's second law, the acceleration of the mass due to this force is equal to the force divided by the mass (provided mass is constant); that is, acceleration=(force)/(mass), or force=(mass)(acceleration). Since acceleration is the second derivative of the displacement or position function, d^2x/dt^2, we can write that

$$-kx - \beta\frac{dx}{dt} + F(t) = M\frac{d^2x}{dt^2}. \tag{4.14}$$

When this equation is rearranged into the form,

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t), \tag{4.15}$$

we have a linear, second-order differential equation for the position function $x(t)$. The equation is homogeneous or nonhomogeneous depending on whether forces other than the spring and damping forces act on the mass.

Before considering specific situations, we show that when masses are suspended vertically from springs, their motion is also governed by equation 4.15, (provided we choose the origin for vertical displacements wisely). To describe the position of the mass M in Figure 4.6 as a function of time t, we need a vertical coordinate system.

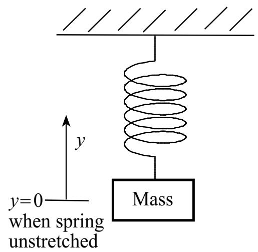


Figure 4.6

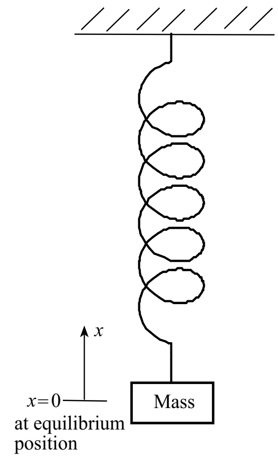


Figure 4.7

There are two natural places to choose the origin. One is to choose $y = 0$ at the position of M when the spring is unstretched. Suppose we do this and choose y as positive upward. When M is a distance y away from the origin, the restoring force of the spring is $-ky$. In addition, if $g = 9.81$ is the acceleration due to gravity, then the force of gravity on M is $-Mg$. In the presence of damping forces or a dashpot, there is a force of the form $-\beta(dy/dt)$, where β is a positive constant. If $F(t)$ represents all other forces acting on M, then the total force on M is $-ky - Mg - \beta(dy/dt) + F(t)$, and Newton's second law for the acceleration of M gives

$$-ky - Mg - \beta\frac{dy}{dt} + F(t) = M\frac{d^2y}{dt^2}.$$

Consequently, the differential equation that determines the position $y(t)$ of M relative to the unstretched position of the spring is

$$M\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = -Mg + F(t). \tag{4.16}$$

The alternative possibility for describing vertical oscillations is to attach M to the spring and slowly lower M until it reaches an equilibrium position. At this position, the restoring force of the spring is exactly equal to the force of gravity on the mass, and the mass, left by itself, will remain motionless. If s is the amount of stretch in the spring at equilibrium, and g is the acceleration due to gravity, then at equilibrium

$$ks - Mg = 0, \qquad \text{where } s > 0 \text{ and } g > 0. \tag{4.17}$$

Suppose we take the equilibrium position as $x = 0$ and x as positive upward (Figure 4.7). When M is a distance x away from its equilibrium position, the restoring force of the spring on M is $k(s - x)$. The force of gravity remains as $-Mg$, and that of the damping force is $-\beta(dx/dt)$. If $F(t)$ accounts for any other forces acting on M, Newton's second law implies that

$$M\frac{d^2x}{dt^2} = k(s - x) - Mg - \beta\frac{dx}{dt} + F(t),$$

or,

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = -Mg + ks + F(t).$$

But according to equation 4.17, $ks - Mg = 0$, and hence

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t). \tag{4.18}$$

This is the differential equation describing the displacement $x(t)$ of M relative to the equilibrium position of M.

Equations 4.16 and 4.18 are both linear second-order differential equations with constant coefficients. The advantage of equation 4.18 is that nonhomogeneity $-Mg$ is absent, and this is simply due to a convenient choice of dependent variable (x as opposed to y). Physically, there are two parts to the spring force $k(s-x)$; a part ks and a part $-kx$. Gravity is always acting on M, and that part ks of the spring force is counteracting gravity in an attempt to restore the spring to its unstretched position. Because these forces always cancel, we might just as well eliminate both of them from our discussion. This leaves $-kx$, and we therefore interpret $-kx$ as the spring force attempting to restore the mass to its equilibrium position.

If we choose equation 4.18 to describe vertical oscillations (and this equation is usually chosen over 4.16), we must remember three things: x is measured from equilibrium, $-kx$ is the spring force attempting to restore M to its equilibrium position, and gravity has been taken into account.

Equation 4.18 for vertical oscillations and equation 4.15 for horizontal oscillations are identical; we have the same differential equation describing either type of oscillation. In both cases, x measures the distance of the mass from its equilibrium position. In the horizontal case, this is from the position of the mass when the spring is unstretched; in the vertical case, this is from the position of the mass when it hangs motionless at the end of the spring.

There are three basic ways to initiate motion. First, we can move the mass away from its equilibrium position and then release it, giving it an initial displacement but no initial velocity. Secondly, we can strike the mass at the equilibrium position, imparting an initial velocity but no initial displacement. And finally, we can give the mass both an initial displacement and an initial velocity. Each of these methods adds two initial conditions to the differential equation.

Undamped, Unforced Vibrations

In the remainder of this section, we begin our studies with undamped ($\beta = 0$), unforced ($F(t) = 0$) vibrations. We begin with two numerical examples, and finish with a general discussion.

Example 4.7 A 2-kilogram mass is suspended vertically from a spring with constant 32 newtons per metre. The mass is raised 10 centimetres above its equilibrium position and then released. If damping is ignored, find the position of the mass as a function of time.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, differential equation 4.18 for the motion $x(t)$ of the mass becomes

$$2\frac{d^2x}{dt^2} + 32x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 16x = 0,$$

along with the initial conditions $x(0) = 1/10$, $x'(0) = 0$. The auxiliary equation $m^2 + 16 = 0$ has solutions $m = \pm 4i$. Consequently,

$$x(t) = C_1 \cos 4t + C_2 \sin 4t.$$

The initial conditions require

$$1/10 = C_1, \quad 0 = 4C_2.$$

Thus,

$$x(t) = \frac{1}{10}\cos 4t \text{ m.}$$

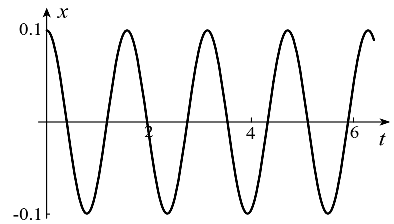


Figure 4.8

A graph of this function (Figure 4.8) illustrates that the mass oscillates about its equilibrium position forever. This is a direct result of the fact that damping has been ignored. The mass oscillates up and down from a position 10 cm above the equilibrium position to a position 10 cm below the equilibrium position. We call 10 cm the **amplitude** of the oscillations. It takes $2\pi/4 = \pi/2$ seconds to complete one full oscillation, and we call this the **period** of the oscillations. The **frequency** of the oscillations is the number of oscillations that take place each second and this is the reciprocal of the period, namely $2/\pi$ Hz (hertz). Oscillations of this kind are called **simple harmonic motion.**•

The spring in this example might be called "loose". We can see this from equation 4.17. Substitution of $M = 2$, $k = 32$, and $g = 9.81$ gives $s = 0.61$ metres; that is, with a 2 kilogram mass suspended at rest from the spring there is a stretch of 61 centimetres. The period of oscillations $\pi/2$ is quite long and the frequency of oscillations is small $2/\pi$. Contrast this with what might be called a stiff spring in the following example.

Example 4.8 The 2-kilogram mass in Example 4.7 is suspended vertically from a spring with constant 3200 newtons per metre. The mass is raised 10 centimetres above its equilibrium position and given an initial velocity of 2 metres per second downward. If damping is ignored, find the position of the mass as a function of time.

Solution The differential equation governing motion is

$$2\frac{d^2x}{dt^2} + 3200x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 1600x = 0,$$

along with the initial conditions $x(0) = 1/10$, $x'(0) = -2$. The auxiliary equation $m^2 + 1600 = 0$ has solutions $m = \pm 40i$. Consequently,

$$x(t) = C_1 \cos 40t + C_2 \sin 40t.$$

The initial conditions require

$$1/10 = C_1, \quad -2 = 40C_2.$$

Thus,

$$x(t) = \frac{1}{10}\cos 40t - \frac{1}{20}\sin 40t \text{ m.}$$

It is more convenient to express this function in the form $A\sin(40t + \phi)$. To find A and ϕ, we set

$$\frac{1}{10}\cos 40t - \frac{1}{20}\sin 40t = A\sin(40t + \phi) = A[\sin 40t\,\cos\phi + \cos 40t\,\sin\phi].$$

Because $\sin 40t$ and $\cos 40t$ are linearly independent functions we equate coefficients to obtain

$$\frac{1}{10} = A\sin\phi, \qquad \frac{-1}{20} = A\cos\phi.$$

When these are squared and added,

$$\frac{1}{100} + \frac{1}{400} = A^2 \qquad \Longrightarrow \qquad A = \frac{\sqrt{5}}{20}.$$

It now follows that ϕ must satisfy the equations

$$\frac{1}{10} = \frac{\sqrt{5}}{20}\sin\phi, \qquad \frac{-1}{20} = \frac{\sqrt{5}}{20}\cos\phi.$$

The smallest positive angle satisfying these is $\phi = 2.03$ radians. The position function of the mass is therefore

$$x(t) = \frac{\sqrt{5}}{20}\sin(40t + 2.03) \text{ m},$$

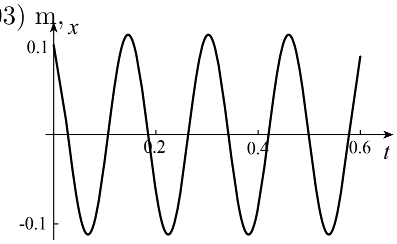


Figure 4.9

a graph of which is shown in Figure 4.9. The amplitude $\sqrt{5}/20$ of the oscillations is slightly larger than that in Example 4.7 due to the fact that the mass was given not only an initial displacement of 10 cm, but also an initial velocity. The spring, with constant $k = 3200$, is one hundred times as tight as that in Example 4.7. The result is a period $\pi/20$ s for the oscillations, one-tenth that in Example 4.7, and ten times as many oscillations per second (frequency is $20/\pi$ Hz).•

General Discussion of Undamped, Unforced Oscillations

When vibrations of a mass M attached to a spring with constant k are unforced and undamped, the differential equation describing displacements $x(t)$ of the mass relative to its equilibrium position is

$$M\frac{d^2x}{dt^2} + kx = 0. \tag{4.19}$$

A general solution of this equation is

$$x(t) = C_1\cos\sqrt{\frac{k}{M}}t + C_2\sin\sqrt{\frac{k}{M}}t. \tag{4.20}$$

This is simple harmonic motion that once again we prefer to write in the form

$$x(t) = A\sin\left(\sqrt{\frac{k}{M}}t + \phi\right), \tag{4.21a}$$

where the amplitude is given by

$$A = \sqrt{C_1^2 + C_2^2}, \tag{4.21b}$$

and angle ϕ is defined by the equations

$$\sin\phi = \frac{C_1}{A} \qquad \text{and} \qquad \cos\phi = \frac{C_2}{A}. \tag{4.21c}$$

Quantity $\sqrt{k/M}$, often denoted by ω, is called the **angular frequency** for the motion. When divided by 2π, $\dfrac{\omega}{2\pi} = \dfrac{\sqrt{k/M}}{2\pi}$, is the **frequency** of the oscillations, the number of oscillations that the mass makes each second. Its inverse $\dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{k/M}}$ is the **period** of the oscillations, the length of time for the mass to make one complete oscillation. Each of these quantities depends only on the mass M and the spring constant k, not on the initial displacement nor the initial velocity of the mass. Notice that frequency increases with k, indicating that stiffer springs produce faster oscillations. Frequency decreases with M so that heavier masses oscillate more slowly than lighter ones.

Initial conditions enter the calculation of the amplitude of the oscillations and angle ϕ. For instance, if the initial displacement and velocity (at time $t = 0$) are x_0 and v_0, then C_1 and C_2 must satisfy the equations

$$x_0 = C_1 \qquad \text{and} \qquad v_0 = \sqrt{\frac{k}{M}}C_2 = \omega C_2.$$

With these initial conditions,

$$x(t) = A\sin(\omega t + \phi), \tag{4.22a}$$

where

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \tag{4.22b}$$

and ϕ is given by

$$\sin\phi = \frac{x_0}{A} \qquad \text{and} \qquad \cos\phi = \frac{v_0/\omega}{A}. \tag{4.22c}$$

The amplitude of the oscillations is constant because without damping, there is no release of the initial energy supplied to the system with the initial displacement and velocity. Angle ϕ is often called the **phase angle** or **angular phase shift**. Quantity $-\phi/\omega$ is called the **phase shift** as it represents the shift in time of the graph of $A\sin(\omega t + \phi)$ along the t-axis relative to that of $A\sin\omega t$.

Alternative forms for solution 4.20 are discussed in Exercise 28.

Damped, Unforced Vibrations

Vibrating mass-spring systems without damping are unrealistic. All vibrations are subject to some degree of damping, and depending on the magnitude of the damping, oscillations either gradually die out, or are completely expunged. Differential equation 4.18 describes the motion of a mass on the end of a spring in the presence of a damping force (with damping constant β) proportional to velocity. When no other forces act on the mass, besides the spring, and gravity for vertical oscillations, the differential equation becomes homogeneous,

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0. \tag{4.23}$$

We shall see that three types of motion occur called *underdamped*, *critically damped*, and *overdamped*. We illustrate with an example of each before giving a general discussion.

Example 4.9 A 50-gram mass is suspended vertically from a very loose spring with constant 5 newtons per metre. The mass is pulled 5 centimetres below its equilibrium position and given velocity 2 metres per second upward. If, during motion, the mass is acted on by a damping force in newtons numerically equal to one-tenth the instantaneous velocity in metres per second, find the position of the mass at any time.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, the differential equation for the position $x(t)$ of the mass is

$$\frac{50}{1000}\frac{d^2x}{dt^2} + \frac{1}{10}\frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 100x = 0,$$

along with the initial conditions $x(0) = -1/20$, $x'(0) = 2$. The auxiliary equation $m^2 + 2m + 100 = 0$ has solutions

$$m = \frac{-2 \pm \sqrt{4 - 400}}{2} = -1 \pm 3\sqrt{11}i.$$

Consequently,

$$x(t) = e^{-t}[C_1 \cos(3\sqrt{11}t) + C_2 \sin(3\sqrt{11}t)].$$

The initial conditions require

$$-\frac{1}{20} = C_1, \qquad 2 = -C_1 + 3\sqrt{11}C_2,$$

from which $C_2 = 13\sqrt{11}/220$. The position of the mass is therefore given by

$$x(t) = e^{-t}\left[-\frac{1}{20}\cos(3\sqrt{11}t) + \frac{13\sqrt{11}}{220}\sin(3\sqrt{11}t)\right] \text{ m}.$$

The graph of this function in Figure 4.10 clearly indicates how the oscillations decrease in time. This is an example of **underdamped motion.**•

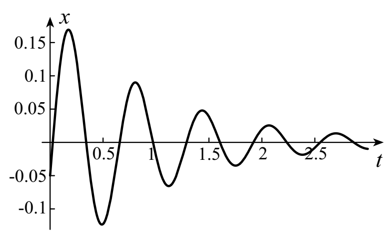


Figure 4.10

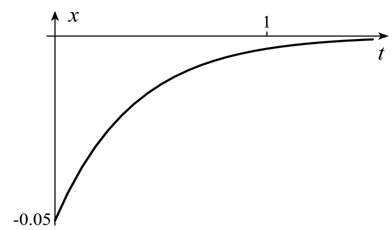


Figure 4.11

Example 4.10 Repeat Example 4.9 if the damping constant is $\beta = 2$.

Solution The differential equation for the position $x(t)$ of the mass is

$$\frac{50}{1000}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 100x = 0,$$

along with the same initial conditions. The auxiliary equation $m^2 + 40m + 100 = 0$ has solutions

$$m = \frac{-40 \pm \sqrt{1600 - 400}}{2} = -20 \pm 10\sqrt{3}.$$

Consequently,

$$x(t) = C_1 e^{(-20+10\sqrt{3})t} + C_2 e^{(-20-10\sqrt{3})t}.$$

The initial conditions require

$$-\frac{1}{20} = C_1 + C_2, \qquad 2 = (-20 + 10\sqrt{3})C_1 + (-20 - 10\sqrt{3})C_2,$$

from which $C_1 = (2\sqrt{3} - 3)/120$ and $C_2 = -(2\sqrt{3} + 3)/120$. The position of the mass is therefore given by

$$x(t) = \left(\frac{2\sqrt{3} - 3}{120}\right) e^{(-20+10\sqrt{3})t} - \left(\frac{2\sqrt{3} + 3}{120}\right) e^{-(20+10\sqrt{3})t} \text{ m}.$$

The graph of this function is shown in Figure 4.11. This is an example of **overdamped motion**; damping is so large that oscillations are completely eliminated. The mass simply returns to the equilibrium position without passing through it.•

Example 4.11 Repeat Example 4.9 if the damping constant is $\beta = 1$.

Solution The differential equation for the position $x(t)$ of the mass is

$$\frac{50}{1000}\frac{d^2x}{dt^2}+\frac{dx}{dt}+5x=0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2}+20\frac{dx}{dt}+100x=0,$$

along with the initial conditions $x(0) = -1/20$, $x'(0) = 2$. The auxiliary equation $m^2 + 20m + 100 = (m+10)^2 = 0$ has a repeated solution $m = -10$. Consequently,

$$x(t) = (C_1 + C_2 t)e^{-10t}.$$

The initial conditions require

$$-\frac{1}{20} = C_1, \qquad 2 = -10C_1 + C_2,$$

from which $C_2 = 3/2$. The position of the mass is therefore given by

$$x(t) = \left(-\frac{1}{20} + \frac{3t}{2}\right)e^{-10t} \text{ m.}$$

The graph of this function is shown in Figure 4.12. This is an example of **critically damped motion**; any smaller value of the damping constant leads to underdamped motion, and any higher value leads to overdamped motion.•

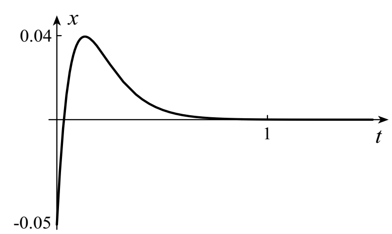


Figure 4.12

General Discussion of Damped, Unforced Motion

We now give a general discussion of differential equation 4.23, clearly delineating values of the parameters M, k, and β that lead to underdamped, critically damped, and overdamped motion. The auxiliary equation associated with

$$M\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=0 \tag{4.23}$$

is the quadratic equation

$$Mm^2+\beta m+k=0, \tag{4.24a}$$

with solutions

$$m=\frac{-\beta\pm\sqrt{\beta^2-4kM}}{2M}. \tag{4.24b}$$

Clearly there are three possibilities depending on the value of $\beta^2 - 4kM$.

Underdamped Motion $\quad \beta^2 - 4kM < 0$

When $\beta^2 - 4kM < 0$, roots 4.24b of the auxiliary equation are complex,

$$m=-\frac{\beta}{2M}\pm\frac{\sqrt{4kM-\beta^2}}{2M}i, \tag{4.25}$$

and a general solution of differential equation 4.23 is

$$x(t)=e^{-\beta t/(2M)}\left[C_1\cos\frac{\sqrt{4kM-\beta^2}}{2M}t+C_2\sin\frac{\sqrt{4kM-\beta^2}}{2M}t\right]. \tag{4.26}$$

If we set $\omega = \dfrac{\sqrt{4kM-\beta^2}}{2M}$, then

$$x(t)=e^{-\beta t/(2M)}(C_1\cos\omega t+C_2\sin\omega t). \tag{4.27}$$

Earlier, we indicated how to express the sine and cosine terms in the form $\sin(\omega t + \phi)$, so that a simplified expression for underdamped oscillations is

$$x(t) = Ae^{-\beta t/(2M)} \sin(\omega t + \phi). \tag{4.28}$$

The presence of the exponential $e^{-\beta t/(2M)}$ before the trigonometric function indicates that we have oscillations that gradually die out. Except possibly for the starting value and initial slope, a typical graph of this function is shown in Figure 4.13. It is contained between the curves $x = \pm Ae^{-\beta/(2M)}$. Motion is not periodic, but the time between successive maxima, or between successive minima, or between successive passes through the equilibrium position when going in the same direction, are all the same. This is often called the **quasi-period** of underdamped motion. It is

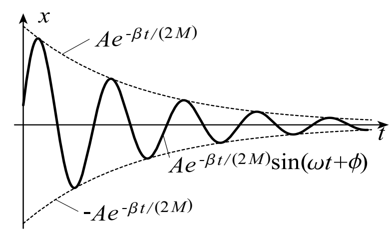


Figure 4.13

$$\frac{2\pi}{\omega} = \frac{2\pi}{\dfrac{\sqrt{4kM - \beta^2}}{2M}} = \frac{2\pi}{\sqrt{\dfrac{k}{M} - \dfrac{\beta^2}{4M^2}}}.$$

Since $\dfrac{2\pi}{\sqrt{k/M}}$ is the period of the motion when damping is absent, the quasi-period is larger than this period, but it approaches $\dfrac{2\pi}{\sqrt{k/M}}$ as $\beta \to 0$. Correspondingly, damping decreases the frequency of oscillations. As damping increases and $\beta^2/(4M^2)$ approaches k/M, the quasi-period becomes indefinitely long and oscillations disappear.

Critically Damped Motion $\quad \beta^2 - 4kM = 0$

This is the limiting case of underdamped motion. Roots 4.24b of the auxiliary equation are real and equal $m = -\beta/(2M)$, and a general solution of differential equation 4.23 is

$$x(t) = (C_1 + C_2 t)e^{-\beta t/(2M)}. \tag{4.29}$$

Damping is so large that oscillations are eliminated and the mass returns from its initial position to the equilibrium position passing through the equilibrium position at most once. This situation forms the division between underdamped motion and overdamped motion (yet to come). Any increase of β results in overdamped motion and any decrease results in underdamped oscillations. Except possibly for starting values and initial slopes, typical graphs of this function are shown in Figure 4.14.

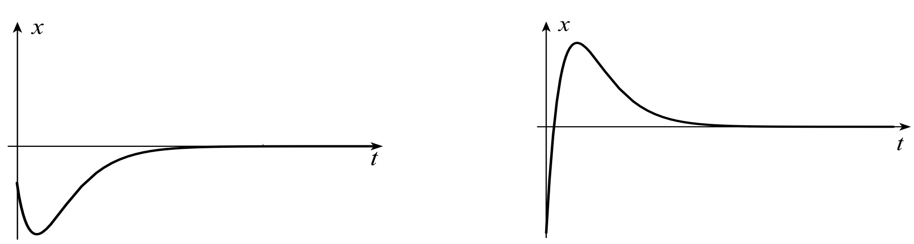


Figure 4.14

Overdamped Motion $\quad \beta^2 - 4kM > 0$

When $\beta^2 - 4kM > 0$, roots 4.24b of the auxiliary equation are real, distinct and negative. A general solution of differential equation 4.23 is

$$x(t) = C_1 e^{(-\beta+\sqrt{\beta^2-4kM})t/(2M)} + C_2 e^{(-\beta-\sqrt{\beta^2-4kM})t/(2M)}. \tag{4.30}$$

Typical graphs of this function are similar to those in Figure 4.14 for critically damped motion.

We now consider further examples of these three possibilities.

Example 4.12 A 100-gram mass is suspended vertically from a spring with constant 5 newtons per metre. The mass is pulled 5 centimetres below its equilibrium position and given velocity 2 metres per second upward. If, during motion, the mass is acted on by a damping force in newtons numerically equal to one-twentieth the instantaneous velocity in metres per second, find the position of the mass at any time.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, the differential equation for the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{1}{20}\frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad 2\frac{d^2x}{dt^2} + \frac{dx}{dt} + 100x = 0,$$

along with the initial conditions $x(0) = -1/20$, $x'(0) = 2$. The auxiliary equation $2m^2 + m + 100 = 0$ has solutions

$$m = \frac{-1 \pm \sqrt{1-800}}{4} = \frac{-1 \pm \sqrt{799}i}{4}.$$

Consequently,

$$x(t) = e^{-t/4}[C_1 \cos(\sqrt{799}t/4) + C_2 \sin(\sqrt{799}t/4)].$$

The initial conditions require

$$-\frac{1}{20} = C_1, \qquad 2 = -\frac{C_1}{4} + \frac{\sqrt{799}C_2}{4},$$

from which $C_2 = 159\sqrt{799}/15980$. The position of the mass is therefore given by

$$x(t) = e^{-t/4}\left[-\frac{1}{20}\cos\left(\frac{\sqrt{799}t}{4}\right) + \frac{159\sqrt{799}}{15980}\sin\left(\frac{\sqrt{799}t}{4}\right)\right] \text{ m}.$$

Using the technique suggested in Example 4.8, we can rewrite the displacement in the form

$$x(t) = Ae^{-t/4}\sin\left(\frac{\sqrt{799}t}{4} + \phi\right),$$

where

$$A = \sqrt{\left(-\frac{1}{20}\right)^2 + \left(\frac{159\sqrt{799}}{15980}\right)^2} \approx 0.285661.$$

The graph of these underdamped oscillations is shown in Figure 4.15. Oscillations are bounded by the curves $x = \pm 0.285661e^{-t/4}$, shown dotted.•

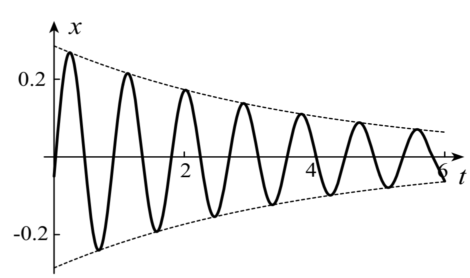


Figure 4.15

Example 4.13 A 4-kilogram mass is attached to a horizontal spring. The mass moves on a frictionless surface, but a dashpot creates a damping force in newtons equal to ten times the velocity of the mass. What spring constant leads to critically damped motion?

Solution Critically damped motion results when spring constant k, mass $M = 4$, and damping factor $\beta = 10$ are related by $\beta^2 - 4kM = 0$; that is, $100 - 4k(4) = 0$. This implies that $k = 25/4$ N/m.•

Example 4.14 A 2-kilogram mass is suspended vertically from a spring with constant 500 newtons per metre. The mass is pulled 10 centimetres below its equilibrium position and given velocity 5 metres per second downward. A dashpot is attached to the mass creating a damping force in newtons numerically equal to one hundred times the instantaneous velocity in metres per second. Show that motion of the mass is overdamped and that in 1 second the mass is within 1 millimetre of its equilibrium position.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, the initial-value problem for the position $x(t)$ of the mass is

$$2\frac{d^2x}{dt^2} + 100\frac{dx}{dt} + 500x = 0, \qquad x(0) = -\frac{1}{10}, \quad x'(0) = -5.$$

The auxiliary equation $2m^2 + 100m + 500 = 2(m^2 + 50m + 250) = 0$ has solutions

$$m = \frac{-50 \pm \sqrt{2500 - 1000}}{2} = -25 \pm 5\sqrt{15}.$$

With real roots, motion is overdamped and the position function is of the form

$$x(t) = C_1 e^{(-25+5\sqrt{15})t} + C_2 e^{(-25-5\sqrt{15})t}.$$

The initial conditions require

$$-\frac{1}{10} = C_1 + C_2, \qquad -5 = (-25 + 5\sqrt{15})C_1 - (25 + 5\sqrt{15})C_2.$$

These can be solved for

$$C_1 = -\frac{\sqrt{15}+1}{20}, \qquad C_2 = \frac{\sqrt{15}-1}{20}.$$

The position of the mass is therefore given by

$$x(t) = -\left(\frac{\sqrt{15}+1}{20}\right) e^{(-25+5\sqrt{15})t} + \left(\frac{\sqrt{15}-1}{20}\right) e^{(-25-5\sqrt{15})t} \text{ m}.$$

If we set $t = 1$, we obtain the position of the mass after one second,

$$x(1) = -\left(\frac{\sqrt{15}+1}{20}\right) e^{(-25+5\sqrt{15})} + \left(\frac{\sqrt{15}+1}{20}\right) e^{(-25-5\sqrt{15})} = -0.000870 \text{ m};$$

that is, the mass is 0.87 millimetres from the equilibrium position.•

Vibrating Mass-Spring Systems With External Forces

So far in this section we have considered mass-spring systems with damping forces, and in the case of vertical oscillations, gravity is also a consideration. With only these forces, the differential equation describing motion is homogeneous. Problems become more interesting, and more widely applicable, when other forces are taken into consideration. In particular, periodic forcing functions lead to *resonance*.

When all other forces acting on the mass in a damped mass-spring system are grouped together into one term denoted by $F(t)$, the differential equation describing motion is

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t). \tag{4.31}$$

We consider various possibilities for $F(t)$. To begin with, you may have noticed that in every example of masses sliding along horizontal surfaces (Figure 4.16), we have ignored friction between the mass and the surface. Suppose we now take it into account. If the coefficient of kinetic friction between the mass and surface is μ, then the force of friction retarding motion has magnitude μMg where $g > 0$ is the acceleration due to gravity. Entering this force into differential equation 4.31 for all time is a problem due to the difficulty in specifying the direction of the force. Certainly we can say that friction is always in a direction opposite to velocity, and we can represent it in the form $-\mu Mg\dfrac{v}{|v|}$, but entering this into equation 4.31 destroys linearity of the equation. The quotient $-v/|v|$ has values ± 1 depending on whether v is negative or positive; it determines the direction of the frictional force. When v is positive, friction is negative (to the left), and when v is negative, friction is positive (to the right). What this means is that each time the mass changes direction, the differential equation must be reconstituted with the appropriate sign attached to μMg. The following example is an illustration.

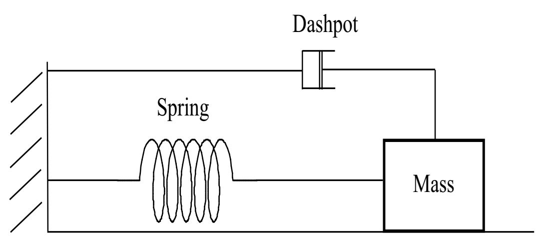


Figure 4.16

Example 4.15 A 1-kilogram mass, attached to a spring with constant 16 newtons per metre, slides horizontally along a surface where the coefficient of kinetic friction between surface and mass is $\mu = 1/10$. Motion is initiated by pulling the mass 10 centimetres to the right of its equilibrium position and giving it velocity 1 metre per second to the left. If any damping forces are negligible, find the point where the mass comes to an instantaneous stop for the second time.

Solution While the mass is travelling to the left for the first time, the force of friction is to the right, and therefore the initial-value problem for its position during this time is

$$\frac{d^2x}{dt^2} + 16x = \left(\frac{1}{10}\right)(1)g, \qquad x(0) = \frac{1}{10}, \quad x'(0) = -1,$$

where $g = 9.81$. Since the auxiliary equation $m^2 + 16 = 0$ has roots $m = \pm 4i$, a general solution of the associated homogeneous differential equation is $x_h(t) = C_1\cos 4t + C_2\sin 4t$. It is easy to spot that a particular solution of the nonhomogeneous equation is $x_p(t) = g/160$, and therefore a general solution of the nonhomogeneous differential equation is

$$x(t) = C_1\cos 4t + C_2\sin 4t + \frac{g}{160}.$$

The initial conditions require

$$\frac{1}{10} = C_1 + \frac{g}{160}, \qquad -1 = 4C_2.$$

Hence,

$$x(t) = \left(\frac{1}{10} - \frac{g}{160}\right)\cos 4t - \frac{1}{4}\sin 4t + \frac{g}{160}\ \text{m}.$$

This represents the position of the mass only while it is travelling to the left for the first time. To determine the time and place at which the mass stops moving to the left, we set the velocity equal to zero,

$$0 = \frac{dx}{dt} = -4\left(\frac{1}{10} - \frac{g}{160}\right)\sin 4t - \cos 4t.$$

This equation can be simplified to

$$\tan 4t = \frac{40}{g-16},$$

solutions of which are

$$t = \frac{1}{4}\,\mathrm{Tan}^{-1}\left(\frac{40}{g-16}\right) + \frac{n\pi}{4},$$

where n is an integer. The only acceptable solution is the smallest positive one, and this occurs for $n = 1$, giving $t = 0.431082$ s. The position of the mass at this time is $x(0.431082) = -0.191663$ m. The mass will move from this position if the spring force is sufficient to overcome the force of static friction. Let us suppose that the coefficient of static friction is $\mu_s = 1/5$. This means that the smallest force necessary for the mass to move has magnitude $(1/5)(1)(9.81) = 1.962$ N. Since the spring force at the first stopping position is $0.191663(16) = 3.06661$ N, it is more than enough to overcome the force of static friction.

For the return trip to the right, friction is to the left, and therefore the initial-value problem for position is

$$\frac{d^2x}{dt^2} + 16x = -\frac{g}{10}, \qquad x(0) = -0.191663, \quad x'(0) = 0.$$

For simplicity, we have reinitialized time $t = 0$ to commencement of motion to the right (see Exercise 47 for the analysis without reinitializing time). A general solution of this differential equation is

$$x(t) = C_3\cos 4t + C_4\sin 4t - \frac{g}{160}.$$

The initial conditions require

$$-0.191663 = C_3 - \frac{g}{160}, \qquad 0 = 4C_4.$$

Thus,

$$x(t) = \left(\frac{g}{160} - 0.191663\right)\cos 4t - \frac{g}{160}\ \text{m}.$$

The mass comes to rest when

$$0 = \frac{dx}{dt} = -4\left(\frac{g}{160} - 0.191663\right)\sin 4t,$$

solutions of which are given by $t = n\pi/4$ where n is an integer. The smallest positive value is $t = \pi/4$ and the position of the mass at this time is

$$x(\pi/4) = \left(\frac{g}{160} - 0.191663\right)\cos\pi - \frac{g}{160} = 0.069038\ \text{m};$$

that is, the mass is 6.9 cm to the right of the equilibrium position. The spring force is still sufficient to overcome the force of friction and the mass will again move to the left.•

Periodic Forcing Functions and Resonance

We now consider the application of periodic forcing functions to masses on the ends of springs. When an external force $F\sin\omega t$, where $F > 0$ and $\omega > 0$ are constants, acts on the mass in a mass-spring system, differential equation 4.31 describing motion becomes

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F\sin\omega t. \tag{4.32}$$

We begin discussions with systems that have no damping, somewhat unrealistic perhaps, but essential ideas are not obscured by intensive calculations. The next example introduces the general discussion to follow.

Example 4.16 A 2-kilogram mass is suspended from a spring with constant 128 newtons per metre. It is pulled 4 centimetres above its equilibrium position and released. An external force $3\sin\omega t$ newtons acts on the mass during its motion. If damping is negligible, find the position of the mass as a function of time.

Solution The initial-value problem for position of the mass is

$$2\frac{d^2x}{dt^2} + 128x = 3\sin\omega t, \qquad x(0) = 1/25, \quad x'(0) = 0.$$

Because the auxiliary equation $2m^2+128 = 0$ has solutions $m = \pm 8i$, a general solution of the associated homogeneous differential equation is $x_h(t) = C_1\cos 8t + C_2\sin 8t$. Undetermined coefficients suggests a particular solution of the form $x_p(t) = A\sin\omega t + B\cos\omega t$. Substitution into the differential equation leads to $x_p(t) = [3/(128-2\omega^2)]\sin\omega t$. Thus, a general solution of the nonhomogeneous differential equation is

$$x(t) = C_1\cos 8t + C_2\sin 8t + \frac{3}{128-2\omega^2}\sin\omega t.$$

The initial conditions require

$$\frac{1}{25} = C_1, \qquad 0 = 8C_2 + \frac{3\omega}{128-2\omega^2}.$$

Thus, the position of the mass at any time is

$$x(t) = \frac{1}{25}\cos 8t + \frac{3\omega}{16(\omega^2-64)}\sin 8t + \frac{3}{2(64-\omega^2)}\sin\omega t \text{ m}.$$

But if $\omega = 8$ the last two terms have vanishing denominators. We should have made allowances for this when determining the particular solution. When $\omega = 8$, the right side of the differential equation is a part of $x_h(t)$, and therefore we should take $x_p(t) = t(A\sin 8t + B\cos 8t)$. Substitution into the differential equation leads to $x_p(t) = -(3t/32)\cos 8t$, and therefore a general solution of the differential equation with nonhomogeneity $3\sin 8t$ is

$$x(t) = C_1\cos 8t + C_2\sin 8t - \frac{3t}{32}\cos 8t.$$

The initial conditions require

$$\frac{1}{25} = C_1, \qquad 0 = 8C_2 - \frac{3}{32}.$$

The position of the mass when the forcing function is $3\sin 8t$ is

$$x(t) = \frac{1}{25}\cos 8t + \frac{3}{256}\sin 8t - \frac{3t}{32}\cos 8t \text{ m}.$$

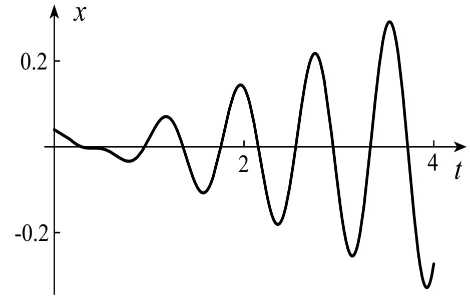


Figure 4.17

A graph of this function is shown in Figure 4.17. The last term in the solution has led to oscillations that become unbounded. This is a direct result of the fact that when $\omega = 8$, the frequency of the forcing term is equal to the frequency at which the system would oscillate were no forcing term present (the so-called **natural frequency** of the system). (Think of this as similar to a parent pushing a child on a swing. Every other time the swing begins its downward motion, the parent applies a force, resulting in the child going higher and higher. The parent applies the force at the same frequency as the motion of the swing.)•

This phenomenon of ever increasing oscillations due to a forcing function with the same frequency as the natural frequency of the system is known as **resonance**. Because the system is undamped, we refer to this as **undamped resonance**.

Let us discuss resonance for the general undamped mass-spring system. When a periodic force $A\sin\omega t$ is applied to an undamped mass-spring system, the differential equation describing motion is

$$M\frac{d^2x}{dt^2} + kx = A\sin\omega t. \tag{4.33}$$

When $\omega \neq \sqrt{k/M}$, the natural frequency of the system and the forcing frequency are different. A general solution of the differential equation takes the form

$$x(t) = C_1\cos(\sqrt{k/M}t) + C_2\sin(\sqrt{k/M}t) + \frac{A}{k - M\omega^2}\sin\omega t, \tag{4.34}$$

and there is nothing untoward about oscillations. When $\omega = \sqrt{k/M}$, so that the forcing frequency is identical to the natural frequency of the undamped system, differential equation 4.33 takes the form

$$M\frac{d^2x}{dt^2} + kx = A\sin(\sqrt{k/M}t), \tag{4.35}$$

In this case the general solution

$$x(t) = C_1\cos(\sqrt{k/M}t) + C_2\sin(\sqrt{k/M}t) - \frac{At}{2\sqrt{kM}}\cos(\sqrt{k/M}t) \tag{4.36}$$

exhibits undamped resonance.

Resonance also occurs in damped systems, but there is a difference; oscillations can be large depending on the degree of damping and the forcing frequency, but they cannot become unbounded. Differential equation 4.32 describes motion of a damped mass-spring system in the presence of a periodic forcing function. Equations 4.26–4.29 define general solutions of the associated homogeneous equation, and it is clear that none of these solutions contain the nonhomogeneity $A\sin\omega t$ for any ω. To put it another way, in the presence of damping, simple harmonic motion is not possible, and therefore the system does not have a natural frequency. Resonance as found in undamped systems is therefore not possible. For underdamped motion, however, oscillations can be large, depending on the degree of damping and the frequency of the applied periodic force, and this is again known as resonance, but we call it **damped resonance**. We illustrate in the following example.

Example 4.17 A 1-kilogram mass is at rest, suspended from a spring with constant 65 newtons per metre. Attached to the mass is a dashpot that creates a damping force equal to twice the velocity of the mass whenever the mass is in motion. At time $t = 0$, a vertical force $3\sin\omega t$ begins to act on the mass. Find the position function for the mass. For what value of ω are oscillations largest?

Solution The initial-value problem for the motion of the mass is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 65x = 3\sin\omega t, \qquad x(0) = 0, \quad x'(0) = 0.$$

The auxiliary equation $m^2+2m+65 = 0$ has solutions $m = -1\pm 8i$ so that a general solution of the associated homogeneous differential equation is $x_h(t) = e^{-t}(C_1\cos 8t + C_2\sin 8t)$. A particular solution can be found in the form $x_p(t) = A\sin\omega t + B\cos\omega t$ by undetermined coefficients. The result is

$$x_p(t) = \frac{3(65-\omega^2)}{(65-\omega^2)^2+4\omega^2}\sin\omega t - \frac{6\omega}{(65-\omega^2)^2+4\omega^2}\cos\omega t.$$

A general solution of the nonhomogeneous differential equation is therefore

$$x(t) = e^{-t}(C_1\cos 8t + C_2\sin 8t) + \frac{3}{(65-\omega^2)^2+4\omega^2}\left[(65-\omega^2)\sin\omega t - 2\omega\cos\omega t\right].$$

The initial conditions require

$$0 = C_1 - \frac{6\omega}{(65-\omega^2)^2+4\omega^2}, \qquad 0 = -C_1 + 8C_2 + \frac{3\omega(65-\omega^2)}{(65-\omega^2)^2+4\omega^2}.$$

These give

$$C_1 = \frac{6\omega}{(65-\omega^2)^2+4\omega^2}, \qquad C_2 = \frac{3\omega(\omega^2-63)}{8[(65-\omega^2)^2+4\omega^2]},$$

and the position of the mass is therefore

$$x(t) = \frac{3\omega e^{-t}}{8[(65-\omega^2)^2+4\omega^2]}[16\cos 8t + (\omega^2-63)\sin 8t] + \frac{3}{(65-\omega^2)^2+4\omega^2}\left[(65-\omega^2)\sin\omega t - 2\omega\cos\omega t\right] \text{ m}.$$

The terms involving $\cos 8t$ and $\sin 8t$ are called the **transient** part of the solution, transient because the e^{-t} factor effectively eliminates these terms after a long time. The terms involving $\sin\omega t$ and $\cos\omega t$, not being subjected to such a factor, do not diminish in time. They are called the **steady-state** part of the solution. In Figure 4.18a we have shown the transient solution; Figure 4.18b shows the steady-state solution with the specific choice $\omega = 4$; and Figure 4.18c shows their sum.

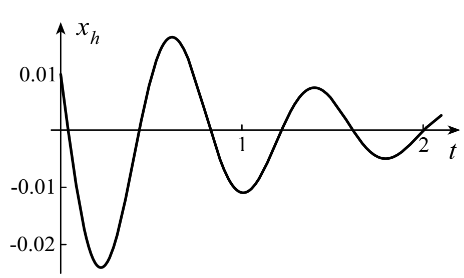


Figure 4.18a

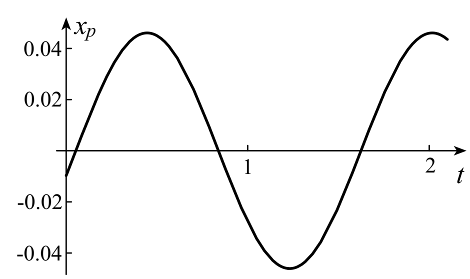


Figure 4.18b

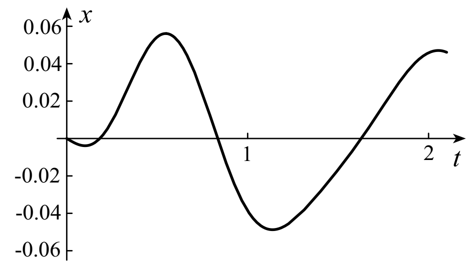


Figure 4.18c

When the forcing frequency is equal to the natural frequency in undamped systems, resonance in the form of unbounded oscillations occurs. Inspection of the above solution indicates that for no value of ω can oscillations become unbounded in this damped system. However, there is a value of ω that makes oscillations largest relative to all other values of ω. In particular, because the transient part of the solution becomes negligible after a sufficiently long time, we are interested in maximizing the amplitude of the steady-state part of the solution. It is the particular solution $x_p(t)$. The amplitude of the oscillations represented by this term is

$$\sqrt{\left[\frac{3(65-\omega^2)}{(65-\omega^2)^2+4\omega^2}\right]^2+\left[\frac{-6\omega}{(65-\omega^2)^2+4\omega^2}\right]^2}=\frac{3}{\sqrt{(65-\omega^2)^2+4\omega^2}};$$

that is, the steady-state solution can be expressed in the form

$$x_p(t)=\frac{3}{\sqrt{(65-\omega^2)^2+4\omega^2}}\sin{(\omega t+\phi)}$$

for some ϕ. To maximize the amplitude we minimize $(65-\omega^2)^2+4\omega^2$. Setting its derivative equal to zero gives

$$0=2(65-\omega^2)(-2\omega)+8\omega,$$

and the only positive solution of this equation is $\omega=3\sqrt{7}$. For this value of ω, the steady-state solution becomes

$$x_p(t)=\frac{3}{16}\sin{(3\sqrt{7}t)}.$$

Maximum oscillations have been realized and the system is said to be in damped resonance. We have shown a graph of this function in Figure 4.19. Compare the scale on the vertical axis in this figure to that in Figure 4.18b where $\omega=4$. We have shown a plot of amplitude versus ω in Figure 4.20.●

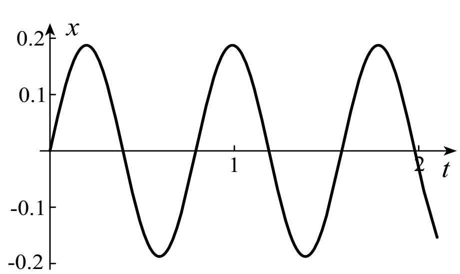


Figure 4.19

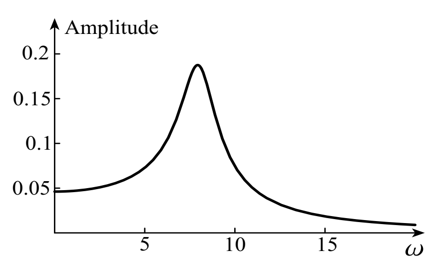


Figure 4.20

We now give a general discussion of damped resonance. When a damped, vibrating mass-spring system is subjected to a sinusoidal input $F\sin{\omega t}$, the differential equation determining displacements of the mass is

$$M\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=F\sin\omega t.$$

Because we are assuming that the motion is underdamped, a general solution of the associated homogeneous equation is given by equation 4.26,

$$x_h(t)=e^{-\beta t/(2M)}\left[C_1\,\cos\frac{\sqrt{4kM-\beta^2}}{2M}t+C_2\,\sin\frac{\sqrt{4kM-\beta^2}}{2M}t\right].$$

A particular solution can be obtained with undetermined coefficients, assuming the solution in the form

$$x_p(t)=B\sin\omega t+D\cos\omega t.$$

When we substitute this into the differential equation, we get

$$M(-\omega^2B\sin\omega t-\omega^2D\cos\omega t)+\beta(\omega B\cos\omega t-\omega D\sin\omega t)$$
$$+k(B\sin\omega t+D\cos\omega t)=F\sin\omega t.$$

We now equate coefficients of terms in $\sin\omega t$ and $\cos\omega t$,

$$-M\omega^2B-\omega\beta D+kB=F,$$
$$-M\omega^2D+\omega\beta B+kD=0.$$

The solution of these equations is

$$B=\frac{F(k-M\omega^2)}{(k-M\omega^2)^2+\beta^2\omega^2},\qquad D=\frac{-\beta\omega F}{(k-M\omega^2)^2+\beta^2\omega^2}.$$

Thus,

$$x_p(t)=\frac{F(k-M\omega^2)}{(k-M\omega^2)^2+\beta^2\omega^2}\sin\omega t-\frac{\beta\omega F}{(k-M\omega^2)^2+\beta^2\omega^2}\cos\omega t.$$

A general solution of the differential equation is $x(t)=x_h(t)+x_p(t)$. We are interested only in the steady-state part of the solution, namely $x_p(t)$. We can write it in the form $x_p(t)=A\sin(\omega t+\phi)$, where the amplitude is given by

$$A=\sqrt{\left[\frac{F(k-M\omega^2)}{(k-M\omega^2)^2+\beta^2\omega^2}\right]^2+\left[\frac{-\beta\omega F}{(k-M\omega^2)^2+\beta^2\omega^2}\right]^2},$$

and this simplifies to $A=\dfrac{F}{\sqrt{(k-M\omega^2)^2+\beta^2\omega^2}}$. Thus,

$$x_p(t)=\frac{F}{\sqrt{(k-M\omega^2)^2+\beta^2\omega^2}}\sin(\omega t+\phi).$$

The quantity

$$Q(\omega)=\frac{1}{\sqrt{(k-M\omega^2)^2+\beta^2\omega^2}} \tag{4.37}$$

is called the **gain factor**, or just plain **gain**, with units of metres per newton. It measures the increase in the amplitude of the motion per newton increase of the applied force. For instance, if the gain is 0.01, then the amplitude of the oscillations increases by 1 centimetre for each newton increase of the applied force. It depends on all four physical quantities in the system, M, k, β, and ω. Our primary interest is in how it depends on ω for fixed M, k, and β. But it is also of interest to see its dependence on β for fixed values of M, k, and ω.

Damped resonance occurs when ω is chosen to maximize the amplitude of $x_p(t)$. This occurs when $Q(\omega)$ is maximized (since F is fixed), and this means when $(k - M\omega^2)^2 + \beta^2\omega^2$ is minimized. Critical values of this function are defined by

$$0 = \frac{d}{d\omega}[(k - M\omega^2)^2 + \beta^2\omega^2 = 2(k - M\omega^2)(-2M\omega) + 2\beta^2\omega.$$

The nontrivial solution of this equation is

$$\omega = \sqrt{\frac{k}{M} - \frac{\beta^2}{2M^2}}.$$

This is the applied frequency for damped resonance. The gain at this frequency is

$$\frac{1}{\sqrt{\left(k - k + \frac{\beta^2}{2M}\right)^2 + \beta^2\left(\frac{k}{M} - \frac{\beta^2}{2M^2}\right)}} = \frac{1}{\beta\sqrt{\frac{k}{M} - \frac{\beta^2}{4M^2}}}.$$

Notice that as β approaches zero, the frequency at damped resonance approaches $\sqrt{k/M}$, the frequency for damped resonance.

Figure 4.21 shows plots of $Q(\omega)$ for four values of β when $k = 1$ and $M = 1$. As β approaches zero, the gain at damped resonance becomes very large.

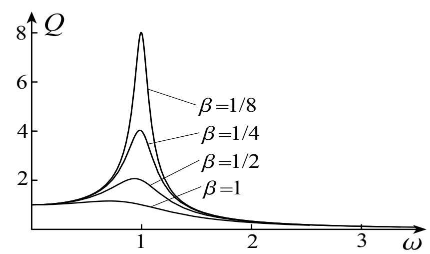


Figure 4.21

EXERCISES 4.3

In Exercises 1–12 find a general solution of the differential equation.

1. $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0$

2. $\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 16y = 0$

3. $\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 41y = 0$

4. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 2y = 0$

5. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 7y = 0$

6. $\frac{d^2y}{dt^2} + 24\frac{dy}{dt} + 144y = 0$

7. $2y'' - 16y' + 32y = -e^{4t}$

8. $y'' + 2y' - 2y = t^2e^{-t}$

9. $y'' + y' - 6y = t + \cos t$

10. $y'' - 4y' + 5y = t\cos t$

11. $y'' + 2y' - 4y = \cos^2 t$

12. $2y'' - 4y' + 3y = \cos t\,\sin 2t$

Undamped, Unforced Vibration Exercises

13. Express the solution in Example 4.8 in the form $x(t) = A\cos(40t - \phi)$.

14. A 1-kilogram mass is suspended vertically from a spring with constant 16 newtons per metre. The mass is pulled 10 centimetres below its equilibrium position and then released. Find the position of the mass, relative to its equilibrium position, at any time if damping is ignored.

15. A 100-gram mass is attached to a spring with constant 100 newtons per metre as in Figure 4.4. The mass is pulled 5 centimetres to the right and released. Find the position of the mass if damping, and friction over the sliding surface, are ignored. Sketch a graph of the position function identifying the amplitude, period, and frequency of the oscillations.

16. Repeat Exercise 15 if motion is initiated by striking the mass, at equilibrium, so as to impart a velocity of 3 metres per second to the left.

17. Repeat Exercise 15 if motion is initiated by pulling the mass 5 centimetres to the right and giving it an initial velocity 3 metres per second to the left.

18. Repeat Exercise 15 if motion is initiated by pulling the mass 5 centimetres to the left and giving it an initial velocity 3 metres per second to the left.

19. (a) A 2-kilogram mass is suspended from a spring with constant 1000 newtons per metre. If the mass is pulled 3 centimetres below its equilibrium position and given a downward velocity of 2 metres per second, find its position thereafter. Sketch a graph of the position function identifying the amplitude, period, and frequency of the oscillations.
(b) Do the initial displacement and velocity affect the amplitude, period, and/or frequency?

20. If the mass in Exercise 19 is quadrupled, how does this affect the period and frequency of the oscillations?

21. If the spring constant in Exercise 19 is quadrupled, how does this affect the period and frequency of the oscillations?

22. When a 2-kilogram mass is set into vertical vibrations on the end of a spring, 3 full oscillations occur each second. What is the spring constant if there is no damping?

23. A mass M is suspended from a spring with constant k. Oscillations are initiated by giving the mass a displacement x_0 and velocity v_0. Show that the position of the mass relative to its equilibrium position, when damping is ignored, can be expressed in the form

$$x(t) = A\sin\left(\sqrt{k/M}\,t + \phi\right),$$

where the amplitude is $A = \sqrt{x_0^2 + Mv_0^2/k}$, and ϕ satisfies

$$\sin\phi = \frac{x_0}{A}, \qquad \cos\phi = \frac{\sqrt{M/k}\,v_0}{A}.$$

24. Use the result of Exercise 23 to show that when the mass on the end of a spring is doubled, the period increases by a factor of $\sqrt{2}$ and the frequency decreases by a factor of $1/\sqrt{2}$.

25. Show the following for oscillations of a mass on the end of a spring when damping is ignored:
(a) Maximum velocity occurs when the mass passes through its equilibrium position. What is the acceleration at this instant?
(b) Maximum acceleration occurs when the mass is at its maximum distance from equilibrium. What is the velocity there?

26. When a spring is suspended vertically, its own weight causes it to stretch. Would this have any effect on our analysis of motion of a mass suspended from the spring?

27. A 100-gram mass is suspended vertically from a spring with constant 40 newtons per metre. The mass is pulled 2 centimetres below its equilibrium position and given an upward velocity of 10 metres per second. Determine:
(a) the position of the mass as a function of time
(b) the amplitude, period, and frequency of the oscillations
(c) all times when the mass has velocity zero
(d) all times when the mass passes through the equilibrium position
(e) all times when the mass has velocity 2 metres per second
(f) all times when the mass is 1 centimetre above the equilibrium position
(g) whether the mass ever has velocity 12 metres per second
(h) the second time the mass is at a maximum height above the equilibrium position.

28. Simple harmonic motion as represented by equation 4.20 can be expressed in alternative forms to 4.21a, namely, $A\sin(\omega t - \phi)$, $A\cos(\omega t + \phi)$, and $A\cos(\omega t - \phi)$. In each case, formula 4.21b for amplitude A is unchanged, only equations 4.21c for angle ϕ change. Show that:

(a) For $A\sin(\omega t - \phi)$,

$$\sin\phi = -\frac{C_1}{A}, \qquad \cos\phi = \frac{C_2}{A}.$$

(b) For $A\cos(\omega t + \phi)$,

$$\sin\phi = -\frac{C_2}{A}, \qquad \cos\phi = \frac{C_1}{A}.$$

(c) For $A\cos(\omega t - \phi)$,

$$\sin\phi = \frac{C_2}{A}, \qquad \cos\phi = \frac{C_1}{A}.$$

29. At time $t = 0$, a mass M is attached to the end of a hanging spring with constant k, and then released. Assuming that damping is negligible, find the subsequent displacement of the mass as a function of time.

30. (a) A cube L metres on each side and with mass M kilograms floats half submerged in water. If it is pushed down slightly and then released, oscillations take place. Use Archimedes' principle to find the differential equation governing these oscillations. Assume no damping forces due to the viscosity of the water.

(b) What is the frequency of the oscillations?

31. A cylindrical buoy 20 centimetres in diameter floats partially submerged with its axis vertical. When it is depressed slightly and released, its oscillations have a period equal to 4 seconds. What is the mass of the buoy?

32. A sphere of radius R floats half submerged in water. It is set into vibration by pushing it down slightly and then releasing it. If y denotes the instantaneous distance of its centre below the surface, show that

$$\frac{d^2y}{dt^2} = \frac{-3g}{2R^3}\left(R^2 y - \frac{y^3}{3}\right),$$

where g is the acceleration due to gravity. Is this a linear differential equation?

Damped Unforced Vibration Exercises

33. A 1-kilogram mass is suspended vertically from a spring with constant 16 newtons per metre. The mass is pulled 10 centimetres below its equilibrium position and then released. Find the position of the mass, relative to its equilibrium position, if a damping force in newtons equal to one-tenth the instantaneous velocity in metres per second acts on the mass.

34. Repeat Exercise 33 if the damping force is equal to ten times the instantaneous velocity.

35. What damping factor creates critically damped motion for the spring and mass in Exercise 33?

36. A 100-gram mass is suspended vertically from a spring with constant 4000 newtons per metre. The mass is pulled 2 centimetres above its equilibrium position and given a downward velocity of 4 metres per second. Find the position of the mass, relative to its equilibrium position, if a dashpot is attached to the mass so as to create a damping force in newtons equal to forty times the instantaneous velocity in metres per second. Does the mass ever pass through the equilibrium position?

37. Repeat Exercise 36 if the mass is given a downward velocity of 10 metres per second.

38. (a) A 1-kilogram mass is suspended vertically from a spring with constant 50 newtons per metre. The mass is pulled 5 centimetres above its equilibrium position and given an upward velocity of 3 metres per second. Find the position of the mass, relative to its equilibrium position, if a dashpot is attached

to the mass so as to create a damping force in newtons equal to fifteen times the instantaneous velocity in metres per second.
(b) Does the mass ever pass through the equilibrium position?
(c) When is the mass 1 centimetre from the equilibrium position?
(d) Sketch a graph of the position function.

39. Repeat Exercise 38 if the initial velocity is 1 metre per second downward.

40. Repeat Exercise 38 if the initial velocity is 3 metres per second downward.

41. (a) A 2-kilogram mass is suspended vertically from a spring with constant 200 newtons per metre. The mass is pulled 10 centimetres above its equilibrium position and given an upward velocity of 5 metres per second. Find the position of the mass, relative to its equilibrium position, if a damping force in newtons equal to four times the instantaneous velocity in metres per second also acts on the mass.
(b) What is the maximum distance the mass attains from equilibrium?
(c) When does the mass first pass through the equilibrium position?

42. (a) A 1-kilogram mass is suspended vertically from a spring with constant 40 newtons per metre. The mass is pulled 5 centimetres below its equilibrium position and released. Find the position of the mass, relative to its equilibrium position, if a dashpot is attached to the mass so as to create a damping force in newtons equal to twice the instantaneous velocity in metres per second. Express the function in the form $Ae^{-at}\sin(\omega t + \phi)$ for appropriate a, A, ω, and ϕ.
(b) Show that the length of time between successive passes through the equilibrium position is constant. What is this time? Twice its value is often called the **quasi period** for overdamped motion? Is it the same as the period of the corresponding undamped system?

43. A mass M is suspended from a spring with constant k. Motion is initiated by giving the mass a displacement x_0 from equilibrium and a velocity v_0. A damping force with constant $\beta > 0$ results in overdamped motion.
(a) Show that if x_0 and v_0 are both positive or both negative, the mass cannot pass through its equilibrium position.
(b) When x_0 and v_0 have opposite signs, it is possible for the mass to pass through the equilibrium position, but it can do so only once. What condition must x_0 and v_0 satisfy for this to happen?

44. A weighing platform has weight W and is supported by springs with combined spring constant k. A package with weight w is dropped on the platform so that the two move together. Find a formula for the maximum value of w so that oscillations do not occur. Assume that there is damping in the motion with constant β.

45. Suppose a mass M is attached to a vertical spring with constant k and damping is increased, taking the system from underdamped motion, through critically damped motion, to overdamped motion. Show that the rate at which the mass returns to its equilibrium position is fastest for critically damped motion. Compare rates for underdamped and overdamped motions.

46. A mass M is suspended from a spring with constant k. Oscillations are initiated by giving the mass a displacement x_0 from equilibrium and a velocity v_0. A damping force with constant $\beta > 0$ results in underdamped motion.
(a) Show that the position of the mass relative to its equilibrium position can be expressed in the form

$$x(t) = Ae^{-\beta t/(2M)}\sin\left(\frac{\sqrt{4kM-\beta^2}}{2M}t + \phi\right),$$

where A and ϕ are constants.
(b) Show that the length of time between successive passes through the equilibrium position is constant. What is this time?
(c) Let t_1, t_2, ..., be the times at which the velocity of the mass is equal to zero (and therefore the times at which $x(t)$ has relative maxima and minima. If x_1, x_2, ..., are the corresponding values of $x(t)$, show that the ratio

$$\frac{x_n}{x_{n+2}} = e^{2\pi\beta/\sqrt{4kM-\beta^2}},$$

is a constant independent of n. The quantity $2\pi\beta/\sqrt{4kM-\beta^2}$ is called the **logarithmic decrement**.

Forced Vibration Exercises

47. Repeat Example 4.15 without reinitializing time for movement to the right.

48. A 0.5-kilogram mass sits on a table attached to a spring with constant 18 newtons per metre (Figure 4.16). The mass is pulled so as to stretch the spring 6 centimetres and then released.

(a) If friction between the mass and the table creates a force of 0.5 newtons that opposes motion, but damping is negligible, show that the differential equation determining motion is

$$\frac{d^2x}{dt^2} + 36x = 1, \quad x(0) = 0.06, \ x'(0) = 0.$$

Assume that the coefficient of static friction is twice the coefficient of kinetic friction.

(b) Find where the mass comes to rest for the first time. Will it move from this position?

49. Repeat Exercise 48 given that the mass is pulled 25 centimetres to the right.

50. A 200-gram mass rests on a table attached to an unstretched spring with constant 5 newtons per metre. The mass is given a velocity of 1/2 metre per second to the right. During the subsequent motion, the coefficient of kinetic friction between mass and table is $\mu_k = 1/4$, but damping is negligible. Where does the mass come to a complete stop? Assume that the coefficient of static friction is $\mu_s = 1/2$.

51. Repeat Exercise 50 if the initial velocity is 2 metres per second.

52. A 100-gram mass is suspended from a spring with constant 4000 newtons per metre. At its equilibrium position, it is suddenly (time $t = 0$) given an upward velocity of 10 metres per second. If an external force $3\cos 100t, \ t \geq 0$ acts on the mass, find its displacement as a function of time. Does resonance occur?

53. Repeat Exercise 52 if the external force is $3\cos 200t$.

54. A vertical spring having constant 64 newtons per metre has a 1-kilogram mass attached to it. An external force $2\sin 4t, \ t \geq 0$ is applied to the mass. If the mass is at rest at its equilibrium position at time $t = 0$, and damping is negligible, find the position of the mass as a function of time. Does resonance occur?

55. Repeat Exercise 54 if the external force is $2\sin 8t$.

56. A mass M is suspended from a vertical spring with constant k. If an external force $F(t) = A\cos\omega t$ is applied to the mass for $t > 0$, find the value of ω that causes resonance.

57. A 200-gram mass suspended vertically from a spring with constant 10 newtons per metre is set into vibration by an external force in newtons given by $4\sin 10t, \ t \geq 0$. During the motion a damping force in newtons equal to 3/2 the velocity on the mass in metres per second acts on the mass. Find the position of the mass as a function of time t.

58. (a) A 1-kilogram mass is motionless, suspended from a spring with constant 100 newtons per metre. A vertical force $2\sin\omega t$ acts on the mass beginning at time $t = 0$. Oscillations are subject to a damping force in newtons equal to twice the velocity in metres per second. Find the position of the mass as a function of time.

(b) What value of ω causes resonance? What is the amplitude of steady-state oscillations for resonance?

59. A mass M is suspended from a spring with constant k. Vertical motion is initiated by an external force $A\cos\omega t$ where A is a positive constant. During the subsequent motion a damping force acts on the mass with damping coefficient β.

(a) Show that the steady-state part of the solution is

$$x_p(t) = \frac{A(k - M\omega^2)}{(k - M\omega^2)^2 + \beta^2\omega^2}\cos\omega t - \frac{A\omega\beta}{(k - M\omega^2)^2 + \beta^2\omega^2}\sin\omega t.$$

(b) Find the value of ω that gives resonance and the resulting amplitude of oscillations.

60. A battery of springs is placed between two sheets of wood, and the structure is placed on a level floor. Equivalent to the springs is a single spring with constant 1000 newtons per metre. A 20 kilogram mass is dropped onto the upper platform, hitting the platform with speed 2 metres per second, and remains attached to the platform thereafter.

(a) Find the position of the mass relative to where it strikes the platform as a function of time. Assume that air drag is 10 times the velocity of the mass.

(b) What is the maximum displacement from where it strikes the platform experienced by the mass?

61. A mass M, attached to a spring with constant k, rests on a horizontal table. At time $t = 0$ it is pulled to the right a distance x_0 and given velocity v_0 to the right. If damping is ignored, but the coefficient of kinetic friction between table and mass is μ, find a formula for the time when the mass comes to an instantaneous stop for the first time.

62. Repeat Exercise 61 if the initial velocity is to the left.

63. A cube 1 metre on each side and with density 1200 kilograms per cubic metre is placed with one of its faces in the surface of a body of water. When the cube is released from this position and sinks, it is acted upon by three forces, gravity, a buoyant force equal to the weight of water displaced by the submerged portion of the cube (Archimedes' principle), and a resistive force equal to twice the speed of the object. Find the depth of the bottom surface of the cube as a function of time from the instant the cube is released until it is completely submerged. Plot a graph of the function.

64. A cube 1 metre on each side and with density 500 kilograms per cubic metre is placed with one of its faces in the surface of a body of water. When the cube is released from this position and sinks, it is acted upon by three forces, gravity, a buoyant force equal to the weight of water displaced by the submerged portion of the cube (Archimedes' principle), and a resistive force equal to twice the speed of the object. Find the depth of the bottom surface of the cube as a function of time. Plot a graph of the function.

65. A cable hangs over a peg, 10 metres on one side and 15 metres on the other. Find the time for it to slide off the peg

(a) if friction at the peg is negligible.

(b) if friction at the peg is equal to the weight of 1 metre of cable.

Answers

1. $y = C_1e^{-3t} + C_2e^{2t}$ **2.** $y = (C_1 + C_2t)e^{4t}$ **3.** $y = e^{-4t}(C_1\cos 5t + C_2\sin 5t)$

4. $y = C_1e^{(-1+\sqrt{3})t} + C_2e^{-(1+\sqrt{3})t}$ **5.** $y = e^{-2t}(C_1\cos\sqrt{3}t + C_2\sin\sqrt{3}t)$ **6.** $y = (C_1 + C_2t)e^{-12t}$

7. $y = (C_1 + C_2t)e^{4t} - \dfrac{t^2}{4}e^{4t}$ **8.** $y = C_1e^{-(1+\sqrt{3})t} + C_2e^{(-1+\sqrt{3})t} - \dfrac{1}{9}(3t^2 + 2)e^{-t}$

9. $y = C_1e^{-3t} + C_2e^{2t} - \dfrac{1}{36}(6t + 1) + \dfrac{1}{50}(\sin t - 7\cos t)$

10. $y = e^{2t}(C_1\cos t + C_2\sin t) + \dfrac{t}{8}(\cos t - \sin t) + \dfrac{1}{16}(\cos t - 2\sin t)$

11. $y = C_1e^{-(1+\sqrt{5})t} + C_2e^{(-1+\sqrt{5})t} - \dfrac{1}{8} + \dfrac{1}{40}(\sin 2t - 2\cos 2t)$

12. $y = e^t\left(C_1\cos\dfrac{t}{\sqrt{2}} + C_2\sin\dfrac{t}{\sqrt{2}}\right) + \dfrac{1}{246}(4\cos 3t - 5\sin 3t) + \dfrac{1}{34}(4\cos t + \sin t)$

13. $\dfrac{\sqrt{5}}{20}\cos(40t + 0.464)$ **14.** $-\dfrac{1}{10}\cos 4t$ m

15. $\dfrac{1}{20}\cos 10\sqrt{10}t$ m, amplitude $= 5$ cm, period $= \dfrac{\sqrt{10}\pi}{50}$ s, frequency $= \dfrac{5\sqrt{10}}{\pi}$ Hz

16. $-\dfrac{3\sqrt{10}}{100}\sin 10\sqrt{10}t$ m, amplitude $= \dfrac{3\sqrt{10}}{100}$ m, period $= \dfrac{\sqrt{10}\pi}{50}$ s, frequency $= \dfrac{5\sqrt{10}}{\pi}$ Hz

17. $\frac{1}{20}\cos 10\sqrt{10}t - \frac{3\sqrt{10}}{100}\sin 10\sqrt{10}t$ m, amplitude $= \frac{\sqrt{115}}{100}$ m, period $= \frac{\sqrt{10}\pi}{50}$ s, frequency $= \frac{5\sqrt{10}}{\pi}$ Hz

18. $-\frac{1}{20}\cos 10\sqrt{10}t - \frac{3\sqrt{10}}{100}\sin 10\sqrt{10}t$ m, amplitude $= \frac{\sqrt{115}}{100}$ m, period $= \frac{\sqrt{10}\pi}{50}$ s, frequency $= \frac{5\sqrt{10}}{\pi}$ Hz

19.(a) $-\frac{3}{100}\cos 10\sqrt{5}t - \frac{\sqrt{5}}{25}\sin 10\sqrt{5}t$ m, amplitude $= \frac{\sqrt{89}}{100}$ m, period $= \frac{\sqrt{5}\pi}{25}$ s, frequency $= \frac{5\sqrt{5}}{\pi}$ Hz (b) Amplitude, but not period or frequency

20. Period doubled, frequency halved **21.** Period halved, frequency doubled **22.** $72\pi^2$ N/m

26. No

27.(a) $-\frac{1}{50}\cos 20t + \frac{1}{2}\sin 20t$ m (b) amplitude $= \frac{\sqrt{626}}{50}$ m, period $= \frac{\pi}{10}$ s, frequency $= \frac{10}{\pi}$ Hz

(c) $\frac{(2n+1)\pi}{40} + \frac{1}{20}\mathrm{Sin}^{-1}\left(\frac{1}{\sqrt{626}}\right)$ s, $n \geq 0$ (d) $\frac{n\pi}{20} + \frac{1}{20}\mathrm{Sin}^{-1}\left(\frac{1}{\sqrt{626}}\right)$ s, $n \geq 0$

(e) $\frac{(2n+1)\pi}{40} + \frac{1}{20}\mathrm{Cos}^{-1}\left(\frac{5}{\sqrt{626}}\right) + \frac{1}{20}\mathrm{Sin}^{-1}\left(\frac{1}{\sqrt{626}}\right)$ s, $n \geq 0$

(f) $\frac{n\pi}{20} + \frac{1}{20}\mathrm{Sin}^{-1}\left(\frac{1}{2\sqrt{626}}\right)$ s, $\frac{(2n+1)\pi}{20} - \frac{1}{20}\mathrm{Sin}^{-1}\left(\frac{1}{2\sqrt{626}}\right)$ s, $n \geq 0$ (g) No

(h) $\frac{7\pi}{40} + \frac{1}{20}\mathrm{Sin}^{-1}\left(\frac{1}{\sqrt{626}}\right)$ s

29. $-\frac{Mg}{k}\cos\sqrt{\frac{k}{M}}t$ m **30.** $\frac{0.705}{\sqrt{L}}$ Hz **31.** 124.9ρ kg, where ρ is the density of the buoy

32. No **33.** $e^{-t/20}\left(-\frac{1}{10}\cos\frac{9\sqrt{79}t}{20} - \frac{\sqrt{79}}{7110}\sin\frac{9\sqrt{79}t}{20}\right)$ m **34.** $\frac{1}{30}(e^{-8t} - 4e^{-2t})$ m **35.** $\beta = 8$

36. $\frac{1}{50}e^{-200t}$ m, No **37.** $\left(\frac{1}{50} - 6t\right)e^{-200t}$ m, $t = 1/300$ s

38.(a) $\frac{1}{20}(14e^{-5t} - 13e^{-10t})$ m (b) No (c) $t = \frac{1}{5}\ln(35 + 2\sqrt{290})$ s

39.(a) $\frac{1}{20}(3e^{-10t} - 2e^{-5t})$ m (b) $t = \frac{1}{5}\ln(3/2)$ s (c) $t = \frac{1}{5}\ln(2\sqrt{10} - 5)$ s

40.(a) $\frac{1}{20}(11e^{-10t} - 10e^{-5t})$ m (b) $t = \frac{1}{5}\ln(11/10)$ s (c) $t = \frac{1}{5}\ln(2\sqrt{170} - 25)$ s

41. $\frac{1}{110}e^{-t}(11\cos 3\sqrt{11}t + 17\sqrt{11}\sin 3\sqrt{11}t)$ m (b) 45.7 cm (c) 0.296 s

42.(a) $\frac{1}{\sqrt{390}}\sin(\sqrt{39}t - 1.73)$ m (b) $\pi/\sqrt{39}$ **43.**(b) $\frac{v_0}{x_0} + \frac{\beta}{2M} < 0$ **44.** $\frac{\beta^2 g}{4k} - W$

46.(b) $\frac{2M\pi}{\sqrt{4kM - \beta^2}}$ **48.**(b) $-1/25$ m, No **49.**(b) $-7/36$ m, Yes **50.** $x = 4.20$ cm

51. $x = -11.8$ cm **52.** $-\frac{1}{1000}\cos 200t + \frac{1}{20}\sin 200t + \frac{1}{1000}\cos 100t$ m, No

53. $\left(\frac{1}{20} + \frac{3t}{40}\right)\sin 200t$ m, Yes **54.** $\frac{1}{24}\sin 4t - \frac{1}{48}\sin 8t$ m, No

55. $\frac{1}{64}\sin 8t - \frac{t}{8}\cos 8t$ m, Yes **56.** $\sqrt{\frac{k}{M}}$

57. $e^{-15t/4}\left[\frac{12}{65}\cos\frac{5\sqrt{23}t}{4} + \frac{20}{13\sqrt{23}}\sin\frac{5\sqrt{23}t}{4}\right] - \frac{4}{65}(3\cos 10t + 2\sin 10t)$ m

58.(a) $\frac{1}{(100-\omega^2)^2 + 4\omega^2}\left\{e^{-t}\left[[4\omega\cos(3\sqrt{11}t) + \frac{2\sqrt{11}\omega(\omega^2 - 98)}{33}\sin(3\sqrt{11}t)\right] + \right.$

$[2(100-\omega^2)\sin\omega t - 4\omega\cos\omega t]\Big\}$ (b) $\omega = 7\sqrt{2}$, Amplitude $= \dfrac{\sqrt{11}}{33}$

59. $\sqrt{\dfrac{k}{M} - \dfrac{\beta^2}{2M^2}},\ \dfrac{2AM}{\beta\sqrt{4kM-\beta^2}}$

60.(a) $e^{-t/4}\left[-\dfrac{g}{50}\cos\dfrac{\sqrt{799}t}{4} + \left(\dfrac{400-g}{50\sqrt{799}}\right)\sin\dfrac{\sqrt{799}t}{4}\right] + g/50$ (b)0.51 m

61. $\sqrt{\dfrac{M}{k}}\mathrm{Tan}^{-1}\left(\dfrac{v_0\sqrt{M/k}}{x_0+\mu Mg/k}\right)$

62. $t = \begin{cases} \sqrt{\dfrac{M}{k}}\mathrm{Tan}^{-1}\left(\dfrac{v_0\sqrt{M/k}}{x_0-\mu Mg/k}\right), & \text{when } x_0 < \mu Mg/k \\ \sqrt{\dfrac{M}{k}}\dfrac{\pi}{2}, & \text{when } x_0 = \mu Mg/k \\ \sqrt{\dfrac{M}{k}}\left[\mathrm{Tan}^{-1}\left(\dfrac{v_0\sqrt{M/k}}{x_0-\mu Mg/k}\right)+\pi\right], & \text{when } x_0 > \mu Mg/k. \end{cases}$

63. $6/5 - \dfrac{e^{-t/1200}}{1000\omega}(1200\omega\cos\omega t + \sin\omega t),\ \omega = \sqrt{1\,199\,999}/1200$

64. $1/2 - \dfrac{e^{-t/500}}{1000\omega}(500\omega\cos\omega t + \sin\omega t),\ \omega = \sqrt{499\,999}/500$

65. (a) 2.59 s (b) 2.80 s

§4.4 Systems of Linear Second-order Differential Equations

In this section, we show how linear algebra can simplify the solution of systems of coupled, linear, second-order, homogeneous, differential equations with constant coefficients. Figure 4.22 shows two masses connected by springs. Following the lead of Section 4.3, we use $x_1(t)$ and $x_2(t)$ to denote distances of the masses from their equilibrium positions when the springs are neither stretched nor compressed. To find the differential equations determining motions of the masses, suppose that the masses are at positions x_1 and x_2 relative to their equilibrium positions as shown in Figure 4.22. Both springs act on M_1. Since the stretch (or compression) in the left spring is x_1, this spring exerts a force $-k_1x_1$ on M_1. The stretch (or compression) in the right spring is represented by the difference $x_2 - x_1$. For example, if M_2 is more to the right of its equilibrium position than M_1, then $x_2 - x_1 > 0$, resulting in a stretch of the right spring; whereas if M_1 is more to the right of its equilibrium position than M_2, then $x_2 - x_1 < 0$, resulting in a compresion of the right spring. Thus, the force of the right spring on M_1 is $k_2(x_2 - x_1)$. If we assume, for the moment, that the masses move along a frictionless surface and damping is negligible, Newton's second law for mass M_1 gives

$$M_1\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1). \tag{4.38a}$$

The only force on M_2 is the right spring, and this force is the same as that due to this spring on M_1, except for direction,

$$M_2\frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1). \tag{4.38b}$$

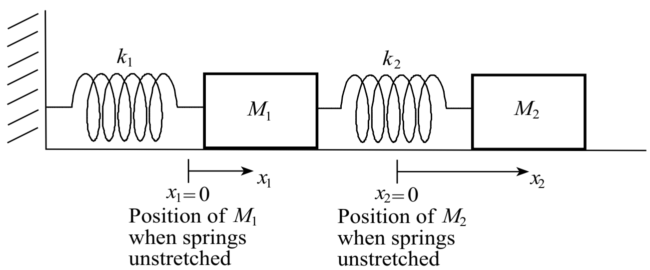


Figure 4.22

In equations 4.38a,b we have a coupled system of linear differential equations for $x_1(t)$ and $x_2(t)$. In a differential equations course, these would be solved using operators, Laplace transforms, and/or matrices. In the operator approach, the equations are decoupled but the result is a fourth-order equation in either $x_1(t)$ or $x_2(t)$. Laplace transforms do not decouple the equations; the transform replaces the system of differential equations with a system of algebraic equations in the transforms of $x_1(t)$ and $x_2(t)$. In the matrix method, two additional variables are introduced $x_3 = dx_1/dt$ and $x_4 = dx_2/dt$, (the velocities of the masses). Second-order system 4.38 is replaced by the first-order system

$$\begin{aligned}
\frac{dx_1}{dt} &= x_3, \\
\frac{dx_2}{dt} &= x_4, \\
\frac{dx_3}{dt} &= -\left(\frac{k_1 + k_2}{M_1}\right)x_1 + \frac{k_2}{M_1}x_2, \\
\frac{dx_4}{dt} &= \frac{k_2}{M_2}x_1 - \frac{k_2}{m_2}x_2.
\end{aligned} \tag{4.39}$$

In matrix form, we have

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2)/M_1 & k_2/M_1 & 0 & 0 \\ k_2/M_2 & -k_2/M_2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Eigenvalues and eigenvectors of the coefficient matrix yield solutions of this vector differential equation, and components then give solutions of the scalar equations. We use eigenpairs in a fundamentally different way here to decouple the differential equations. We can decouple equations 4.38 into second-order differential equations, or we can decouple equations 4.39 into first-order equations. The first option is less work. We illustrate in the following example, where physical constants are somewhat unrealistic, but we have chosen them so that ideas are not obscured by messy calculations.

Example 4.18 Suppose the masses in Figure 4.22 are both one kilogram, and spring constants are $k_1 = 6$ newtons per metre and $k_2 = 4$ newtons per metre. Suppose further that motion is initiated at time $t = 0$ by stretching the left spring 10 centimetres and compressing the right spring 10 centimetres. Find positions of the masses relative to their equilibrium positions.

Solution Equations 4.38a,b become

$$\frac{d^2x_1}{dt^2} = -10x_1 + 4x_2, \qquad \frac{d^2x_2}{dt^2} = 4x_1 - 4x_2.$$

Initially, the left spring is stretched 10 cm so that $x_1(0) = 1/10$. For the right spring to be compressed 10 cm when the left spring is stretched 10 cm, the initial position of M_2 must be 20 cm to the left of its equilibrium position, $x_2(0) = -1/5$. Because the masses are released from these positions, they have zero initial velocities, and we adjoin the following initial conditions to complete the system,

$$x_1(0) = \frac{1}{10}, \qquad x_1'(0) = 0, \qquad x_2(0) = -\frac{1}{5}, \qquad x_2'(0) = 0.$$

In matrix form,

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -10 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Eigenvalues of the matrix are given by

$$0 = \det\begin{pmatrix} -10-\lambda & 4 \\ 4 & -4-\lambda \end{pmatrix} = (\lambda+10)(\lambda+2) - 16 = (\lambda+2)(\lambda+12).$$

Eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -12$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. If $q_1(t)$ and $q_2(t)$ are components of $\mathbf{x}(t)$ with respect to the eigenvector basis, so that $\mathbf{x}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2$, then

$$\frac{d^2}{dt^2}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -12 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Equating entries gives the system of differential equations

$$\frac{d^2q_1}{dt^2} = -2q_1, \qquad \frac{d^2q_2}{dt^2} = -12q_2.$$

The eigenvalue basis has decoupled the system into linear second-order differential equations in $q_1(t)$ and $q_2(t)$. Solutions of these are

$$q_1(t) = C_1 \cos\sqrt{2}t + C_2 \sin\sqrt{2}t, \qquad q_2(t) = C_3 \cos 2\sqrt{3}t + C_4 \sin 2\sqrt{3}t,$$

where C_1, C_2, C_3, and C_4 are constants. To evaluate them, we can transform the initial conditions to the eigenvector basis, or return to the natural basis. We prefer the latter, by writing

$$\mathbf{x}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2 = (C_1 \cos\sqrt{2}t + C_2 \sin\sqrt{2}t)\begin{pmatrix}1\\2\end{pmatrix} + (C_3 \cos 2\sqrt{3}t + C_4 \sin 2\sqrt{3}t)\begin{pmatrix}-2\\1\end{pmatrix}.$$

When we equate components,

$$x_1(t) = C_1 \cos\sqrt{2}t + C_2 \sin\sqrt{2}t - 2C_3 \cos 2\sqrt{3}t - 2C_4 \sin 2\sqrt{3}t,$$
$$x_2(t) = 2C_1 \cos\sqrt{2}t + 2C_2 \sin\sqrt{2}t + C_3 \cos 2\sqrt{3}t + C_4 \sin 2\sqrt{3}t.$$

The initial conditions require

$$\frac{1}{10} = C_1 - 2C_3, \quad 0 = \sqrt{2}C_2 - 4\sqrt{3}C_4, \quad -\frac{1}{5} = 2C_1 + C_3, \quad 0 = 2\sqrt{2}C_2 + 2\sqrt{3}C_4.$$

The solution is $C_1 = -3/50$, $C_3 = -2/25$, and $C_2 = C_4 = 0$. Thus,

$$x_1(t) = -\frac{3}{50}\cos\sqrt{2}t + \frac{4}{25}\cos 2\sqrt{3}t, \quad x_2(t) = -\frac{3}{25}\cos 2\sqrt{3}t - \frac{2}{25}\cos 2\sqrt{3}t.\bullet$$

In the next example, spring constants are more realistic, with the result that calculations are more intensive.

Example 4.19 Suppose the masses in Figure 4.22 are $M_1 = 1$ and $M_2 = 2$ kilograms, and spring constants are $k_1 = 100$ and $k_2 = 500$ newtons per metre. Motion is initiated at time $t = 0$ by stretching the left spring 2 centimetres and compressing the right spring 4 centimetres. Find positions of the masses relative to their equilibrium positions.

Solution Equations 4.38a,b for displacements of the masses are

$$\frac{d^2x_1}{dt^2} = -600x_1 + 500x_2, \qquad 2\frac{d^2x_2}{dt^2} = 500x_1 - 500x_2.$$

Initial conditions are

$$x_1(0) = \frac{1}{50}, \qquad x_1'(0) = 0, \qquad x_2(0) = -\frac{1}{50}, \qquad x_2'(0) = 0.$$

In matrix form,

$$\frac{d^2}{dt^2}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-600 & 500\\250 & -250\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

Eigenvalues, as defined by,

$$0 = \det\begin{pmatrix}-600-\lambda & 500\\250 & -250-\lambda\end{pmatrix} = \lambda^2 + 850\lambda + 25\,000,$$

are $\lambda_1 = -425 + 25\sqrt{249}$ and $\lambda_2 = -425 - 25\sqrt{249}$. Corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix}\sqrt{249}-7\\10\end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix}\sqrt{249}+7\\-10\end{pmatrix}.$$

If $q_1(t)$ and $q_2(t)$ are components of $\mathbf{x}(t)$ with respect to the eigenvector basis, so that $\mathbf{x}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2$, then

$$\frac{d^2}{dt^2}\begin{pmatrix}q_1\\q_2\end{pmatrix} = \begin{pmatrix}-425+25\sqrt{249} & 0\\0 & -425-25\sqrt{249}\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix}.$$

Equating entries gives the system of differential equations

$$\frac{d^2q_1}{dt^2} = (-425 + 25\sqrt{249})q_1, \qquad \frac{d^2q_2}{dt^2} = -(425 + 25\sqrt{249})q_2.$$

The eigenvalue basis has decoupled the system into linear second-order differential equations in $q_1(t)$ and $q_2(t)$. Solutions of these are

$$q_1(t) = C_1 \cos\omega_1 t + C_2 \sin\omega_1 t, \qquad q_2(t) = C_3 \cos\omega_2 t + C_4 \sin\omega_2 t,$$

where $\omega_1 = 5\sqrt{17 - \sqrt{249}}$ and $\omega_2 = 5\sqrt{17 + \sqrt{249}}$. To evaluate the constants C_1, C_2, C_3, and C_4, we can transform the initial conditions to the eigenvector basis, or return to the natural basis. We prefer the latter, by writing

$$\begin{aligned}\mathbf{x}(t) &= q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2 \\ &= (C_1 \cos\omega_1 t + C_2 \sin\omega_1 t)\begin{pmatrix}\sqrt{249} - 7 \\ 10\end{pmatrix} + (C_3 \cos\omega_2 t + C_4 \sin\omega_2 t)\begin{pmatrix}\sqrt{249} + 7 \\ -10\end{pmatrix}.\end{aligned}$$

When we equate components,

$$\begin{aligned}x_1(t) &= (\sqrt{249} - 7)(C_1 \cos\omega_1 t + C_2 \sin\omega_1 t) + (\sqrt{249} + 7)(C_3 \cos\omega_2 t + C_4 \sin\omega_2 t), \\ x_2(t) &= 10(C_1 \cos\omega_1 t + C_2 \sin\omega_1 t) - 10(C_3 \cos\omega_2 t + C_4 \sin\omega_2 t).\end{aligned}$$

The initial conditions require

$$\begin{aligned}\frac{1}{50} &= (\sqrt{249} - 7)C_1 + (\sqrt{249} + 7)C_3, \quad 0 = (\sqrt{249} - 7)\omega_1 C_2 + (\sqrt{249} + 7)\omega_2 C_4, \\ -\frac{1}{50} &= 10C_1 - 10C_3, \quad 0 = 10\omega_1 C_2 - 10\omega_2 C_4.\end{aligned}$$

The solution is $C_1 = (3 - \sqrt{249})/(1000\sqrt{249})$, $C_3 = (3 + \sqrt{249})/(1000\sqrt{249})$, and $C_2 = C_4 = 0$. Thus,

$$\begin{aligned}x_1(t) &= (\sqrt{249} - 7)\left(\frac{3 - \sqrt{249}}{1000\sqrt{249}}\right)\cos\omega_1 t + (\sqrt{249} + 7)\left(\frac{3 + \sqrt{249}}{1000\sqrt{249}}\right)\cos\omega_2 t, \\ &= \left(\frac{\sqrt{249} - 27}{100\sqrt{249}}\right)\cos\omega_1 t + \left(\frac{\sqrt{249} + 27}{100\sqrt{249}}\right)\cos\omega_2 t, \text{ m} \\ x_2(t) &= \left(\frac{3 - \sqrt{249}}{100\sqrt{249}}\right)\cos\omega_1 t - \left(\frac{3 + \sqrt{249}}{100\sqrt{249}}\right)\cos\omega_2 t \text{ m.}\bullet\end{aligned}$$

We have plotted these functions in Figure 4.23. Motion does not appear to be periodic.

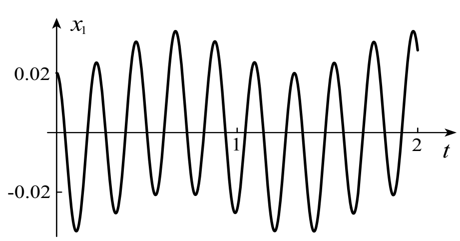


Figure 4.23a

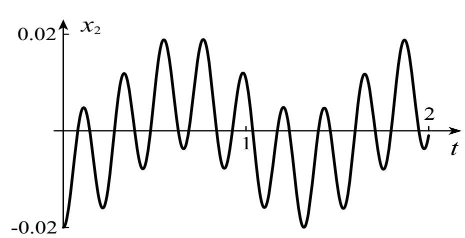


Figure 4.23b

We now add damping to the system in Example 4.18.

Example 4.20 Solve Example 4.18 if each mass is subject to damping equal in magnitude to the velocity of the mass.

Solution With damping taken into account, differential equations for displacements of the masses become

$$\frac{d^2x_1}{dt^2} = -10x_1 + 4x_2 - \frac{dx_1}{dt}, \qquad \frac{d^2x_2}{dt^2} = 4x_1 - 4x_2 - \frac{dx_2}{dt}.$$

Initial conditions remain the same

$$x_1(0) = \frac{1}{10}, \qquad x_1'(0) = 0, \qquad x_2(0) = -\frac{1}{5}, \qquad x_2'(0) = 0.$$

In matrix form,

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -10 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Eigenvalues of the matrix remain $\lambda_1 = -2$ and $\lambda_2 = -12$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. If $q_1(t)$ and $q_2(t)$ are components of $\mathbf{x}(t)$ with respect to the eigenvector basis, so that $\mathbf{x}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2$, then

$$\frac{d^2}{dt^2}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \frac{d}{dt}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -12 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Equating entries gives the system of differential equations

$$\frac{d^2q_1}{dt^2} + \frac{dq_1}{dt} = -2q_1, \qquad \frac{d^2q_2}{dt^2} + \frac{dq_2}{dt} = -12q_2.$$

Solutions of these are

$$q_1(t) = e^{-t/2}\left(C_1 \cos\frac{\sqrt{7}t}{2} + C_2 \sin\frac{\sqrt{7}t}{2}\right), \qquad q_2(t) = e^{-t/2}\left(C_3 \cos\frac{\sqrt{47}t}{2} + C_4 \sin\frac{\sqrt{47}t}{2}\right),$$

where C_1, C_2, C_3, and C_4 are constants. To evaluate them, we can transform the initial conditions to the eigenvector basis, or return to the natural basis. We prefer the latter, by writing

$$\begin{aligned}\mathbf{x}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2 &= e^{-t/2}\left(C_1 \cos\frac{\sqrt{7}t}{2} + C_2 \sin\frac{\sqrt{7}t}{2}\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &\quad + e^{-t/2}\left(C_3 \cos\frac{\sqrt{47}t}{2} + C_4 \sin\frac{\sqrt{47}t}{2}\right)\begin{pmatrix} -2 \\ 1 \end{pmatrix}.\end{aligned}$$

When we equate components,

$$x_1(t) = e^{-t/2}\left(C_1 \cos\frac{\sqrt{7}t}{2} + C_2 \sin\frac{\sqrt{7}t}{2}\right) - 2e^{-t/2}\left(C_3 \cos\frac{\sqrt{47}t}{2} + C_4 \sin\frac{\sqrt{47}t}{2}\right),$$

$$x_2(t) = 2e^{-t/2}\left(C_1 \cos\frac{\sqrt{7}t}{2} + C_2 \sin\frac{\sqrt{7}t}{2}\right) + e^{-t/2}\left(C_3 \cos\frac{\sqrt{47}t}{2} + C_4 \sin\frac{\sqrt{47}t}{2}\right).$$

The initial conditions require

$$\frac{1}{10} = C_1 - 2C_3, \quad 0 = -\frac{C_1}{2} + \frac{\sqrt{7}}{2}C_2 + C_3 - \sqrt{47}C_4,$$

$$-\frac{1}{5} = 2C_1 + C_3, \quad 0 = -C_1 + \sqrt{7}C_2 - \frac{C_3}{2} + \frac{\sqrt{47}}{2}C_4.$$

The solution is $C_1 = -3/50$, $C_2 = -3/(50\sqrt{7})$, $C_3 = -2/25$, and $C_4 = -2/(25\sqrt{47})$. Thus,

$$x_1(t) = -e^{-t/2}\left(\frac{3}{50}\cos\frac{\sqrt{7}t}{2} + \frac{3}{50\sqrt{7}}\sin\frac{\sqrt{7}t}{2}\right) + 2e^{-t/2}\left(\frac{2}{25}\cos\frac{\sqrt{47}t}{2} + \frac{2}{25\sqrt{47}}\sin\frac{\sqrt{47}t}{2}\right),$$
$$x_2(t) = -2e^{-t/2}\left(\frac{3}{50}\cos\frac{\sqrt{7}t}{2} + \frac{3}{50\sqrt{7}}\sin\frac{\sqrt{7}t}{2}\right) - e^{-t/2}\left(\frac{2}{25}\cos\frac{\sqrt{47}t}{2} + \frac{2}{25\sqrt{47}}\sin\frac{\sqrt{47}t}{2}\right).$$

In Figure 4.24, we have added a third spring to a system like that in Figure 4.22. We let $x_1(t)$ and $x_2(t)$ represent positions of the masses relative to the positions that they would occupy were they motionless. Once again we would call this the equilibrium position of the system, but there could be compressions or stretches in the springs. All three are either stretched or all three are compressed; there cannot be both stretches and compressions. We begin our analysis by determining the ratios of stretches (or compressions) at equilibrium. Suppose that s represents the total stretch (or compression) of all three springs at equilibrium. If s_1, s_2, and s_3 are the stretches in the springs, then $s = s_1 + s_2 + s_3$. Since the masses are motionless, we can say that

$$0 = -k_1 s_1 + k_2 s_2, \qquad 0 = -k_2 s_2 + k_3 s_3. \tag{4.40}$$

These imply that $s_1 = k_2 s_2/k_1$ and $s_3 = k_2 s_2/k_3$. If we substitute these into $s = s_1 + s_2 + s_3$, we obtain

$$s = \frac{k_2 s_2}{k_1} + s_2 + \frac{k_2 s_2}{k_3},$$

and this can be solved for

$$s_2 = \frac{s}{k_2/k_1 + 1 + k_2/k_3} = \frac{k_1 k_3 s}{k_1 k_2 + k_2 k_3 + k_3 k_1}.$$

This gives

$$s_1 = \frac{k_2}{k_1}\left(\frac{k_1 k_3 s}{k_1 k_2 + k_2 k_3 + k_3 k_1}\right) = \frac{k_2 k_3 s}{k_1 k_2 + k_2 k_3 + k_3 k_1},$$
$$s_3 = \frac{k_2}{k_3}\left(\frac{k_1 k_3 s}{k_1 k_2 + k_2 k_3 + k_3 k_1}\right) = \frac{k_1 k_2 s}{k_1 k_2 + k_2 k_3 + k_3 k_1}.$$

Thus, the stretches (or compressions) in the springs at equilibrium are in the ratios $k_2 k_3 : k_1 k_3 : k_1 k_2$.

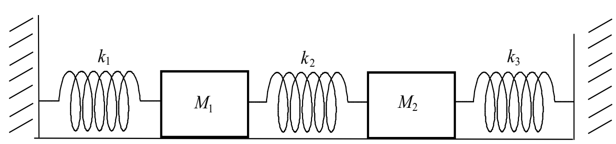


Figure 4.24

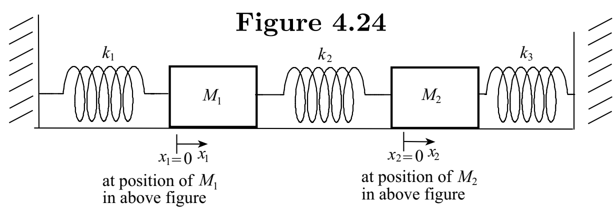


Figure 4.25

We now derive the differential equations for positions $x_1(t)$ and $x_2(t)$ of the masses relative to their equilibrium positions (Figure 4.25). Suppose that s_1, s_2, and s_3 represent the stretches in the springs at equilibrium. When the masses are at positions x_1 and x_2, the stretch in the left spring is $s_1 + x_1$ so that the force of this spring on M_1 is $-k_1(s_1 + x_1)$.

Since the stretch in the middle spring is $s_2+x_2-x_1$, its force on mass M_1 is $k_2(s_2+x_2-x_1)$. Finally, the stretch in the right spring is s_3-x_2, it exerts a force $k_3(s_3-x_2)$ on M_2. Newton's second law applied to the two masses gives

$$M_1\frac{d^2x_1}{dt^2} = -k_1(s_1+x_1)+k_2(s_2+x_2-x_1),$$
$$M_2\frac{d^2x_2}{dt^2} = -k_2(s_2+x_2-x_1)+k_3(s_3-x_2).$$

When we take equations 4.40 into account, these simplify to

$$M_1\frac{d^2x_1}{dt^2} = -k_1x_1+k_2(x_2-x_1), \tag{4.41a}$$
$$M_2\frac{d^2x_2}{dt^2} = -k_2(x_2-x_1)-k_3x_2. \tag{4.41b}$$

Example 4.21 Two 100-gram masses are joined by weak springs as shown in Figure 4.24, where $k_1 = 10$, $k_2 = 20$, and $k_3 = 10$ newtons per metre. At equilibrium, the outside springs are stretched 24 centimetres and the inner spring is stretched 12 centimetres. If mass M_1 is pulled 10 centimetres to the left and mass M_2 is pulled 10 centimetres to the right, and both masses are then released, find their subsequent positions.

Solution Displacements $x_1(t)$ and $x_2(t)$ of the masses from their equilibrium positions must satisfy equations 4.41,

$$\frac{1}{10}\frac{d^2x_1}{dt^2} = -10x_1+20(x_2-x_1), \qquad \frac{1}{10}\frac{d^2x_2}{dt^2} = -20(x_2-x_1)-10x_2.$$

The initial conditions are

$$x_1(0) = -\frac{1}{10}, \qquad x_1'(0) = 0, \qquad x_2(0) = \frac{1}{10}, \qquad x_2'(0) = 0.$$

In matrix form,

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -300 & 200 \\ 200 & -300 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Eigenvalues of the matrix are given by

$$0 = \det\begin{pmatrix} -300-\lambda & 200 \\ 200 & -300-\lambda \end{pmatrix} = (\lambda+300)^2 - 40\,000.$$

Eigenvalues are $\lambda_1 = -100$ and $\lambda_2 = -500$. Corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. If $q_1(t)$ and $q_2(t)$ are components of $\mathbf{x}(t)$ with respect to the eigenvector basis, so that $\mathbf{x}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2$, then

$$\frac{d^2}{dt^2}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -100 & 0 \\ 0 & -500 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Equating entries gives the system of differential equations

$$\frac{d^2q_1}{dt^2} = -100q_1, \qquad \frac{d^2q_2}{dt^2} = -500q_2.$$

Once again, the eigenvalue basis has decoupled the system into linear second-order differential equations in $q_1(t)$ and $q_2(t)$. Solutions of these are

$$q_1(t) = C_1\cos 10t + C_2\sin 10t, \qquad q_2(t) = C_3\cos 10\sqrt{5}t + C_4\sin 10\sqrt{5}t.$$

To evaluate constants, we return to the natural basis

$$\mathbf{x}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2 = (C_1 \cos 10t + C_2 \sin 10t)\begin{pmatrix}1\\1\end{pmatrix} + (C_3 \cos 10\sqrt{5}t + C_4 \sin 10\sqrt{5}t)\begin{pmatrix}1\\-1\end{pmatrix}.$$

When we equate components,

$$x_1(t) = C_1 \cos 10t + C_2 \sin 10t + C_3 \cos 10\sqrt{5}t + C_4 \sin 10\sqrt{5}t,$$
$$x_2(t) = C_1 \cos 10t + C_2 \sin 10t - C_3 \cos 10\sqrt{5}t - C_4 \sin 10\sqrt{5}t.$$

The initial conditions require

$$-\frac{1}{10} = C_1 + C_3, \quad 0 = 10C_2 + 10\sqrt{5}C_4, \quad \frac{1}{10} = C_1 - C_3, \quad 0 = 10C_2 - 10\sqrt{5}C_4.$$

The solution is $C_1 = C_2 = C_4 = 0.$ and $C_3 = -1/10$. Thus,

$$x_1(t) = -\frac{1}{10}\cos 10\sqrt{5}t \text{ m}, \qquad x_2(t) = \frac{1}{10}\cos 10\sqrt{5}t \text{ m}.$$

The masses move in unison; their displacements are opposite in sign. When M_1 moves to the left, M_2 moves to the right, and vice versa.•

Suppose that the masses in Figure 4.22 are turned to the vertical position as in Figure 4.26. If s_1 and s_2 are the stretches in the springs at equilibrium, then,

$$\begin{aligned} 0 &= -k_1 M_1 + k_2 s_2 + M_1 g, \\ 0 &= -k_2 s_2 + m_2 g. \end{aligned} \tag{4.42}$$

When the masses are at positions x_1 and x_2 relative to their equilibrium positions, the stretch in the upper spring is $s_1 - x_1$ so that the force of this spring on M_1 is $k_1(s_1 - x_1)$. Since the stretch in the lower spring is $s_2 + x_1 - x_2$, its force on mass M_1 is $-k_2(s_2 + x_1 - x_2)$. When gravity is also taken into account, Newton's second law for displacements of the masses gives

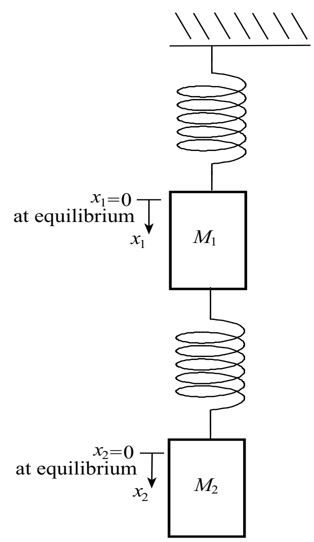


Figure 4.26

$$M_1 \frac{d^2 x_1}{dt^2} = -M_1 g + k_1(s_1 - x_1) - k_2(s_2 + x_1 - x_2),$$
$$M_2 \frac{d^2 x_2}{dt^2} = -M_2 g + k_2(s_2 + x_1 - x_2).$$

Due to equilibrium conditions 4.42, these simplify to

$$M_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1), \tag{4.43a}$$
$$M_2 \frac{d^2 x_2}{dt^2} = -k_2(x_2 - x_1). \tag{4.43b}$$

These are identical to equations 4.38. This was what we found in Section 4.3 for a single mass-spring system. The same differential equation describes vertical and horizontal vibrations as long as displacement is measured relative to the equilibrium position.

Example 4.22 Suppose the masses in Figure 4.26 are $M_1 = 2$ and $M_2 = 1$ kilogram, and spring constants are $k_1 = 100$ and $k_2 = 50$ newtons per metre. The 2 kilogram mass is lifted 10 centimetres above its equilibrium position and released, while the other mass is simultaneously held at its equilibrium position and given velocity 2 metres per second downward. Find the positions of the masses as functions of time.

Solution Displacements $x_1(t)$ and $x_2(t)$ of the masses from their equilibrium positions must satisfy equations 4.43,

$$2\frac{d^2x_1}{dt^2} = -100x_1 + 50(x_2 - x_1), \qquad 1\frac{d^2x_2}{dt^2} = -50(x_2 - x_1).$$

The initial conditions are

$$x_1(0) = \frac{1}{10}, \qquad x_1'(0) = 0, \qquad x_2(0) = 0, \qquad x_2'(0) = -2.$$

In matrix form,

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Eigenvalues of the matrix are given by

$$0 = \det\begin{pmatrix} -75-\lambda & 25 \\ 50 & -50-\lambda \end{pmatrix} = (\lambda + 75)(\lambda + 50) - 1250.$$

Eigenvalues are $\lambda_1 = -25$ and $\lambda_2 = -100$. Corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. If $q_1(t)$ and $q_2(t)$ are components of $\mathbf{x}(t)$ with respect to the eigenvector basis, so that $\mathbf{x}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2$, then

$$\frac{d^2}{dt^2}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -25 & 0 \\ 0 & -100 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Equating entries gives the system of differential equations

$$\frac{d^2q_1}{dt^2} = -25q_1, \qquad \frac{d^2q_2}{dt^2} = -100q_2.$$

Solutions of these are

$$q_1(t) = C_1\cos 5t + C_2\sin 5t, \qquad q_2(t) = C_3\cos 10t + C_4\sin 10t.$$

To evaluate constants, we return to the natural basis

$$\mathbf{x}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2 = (C_1\cos 5t + C_2\sin 5t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + (C_3\cos 10t + C_4\sin 10t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

When we equate components,

$$\begin{aligned} x_1(t) &= C_1\cos 5t + C_2\sin 5t + C_3\cos 10t + C_4\sin 10t, \\ x_2(t) &= 2(C_1\cos 5t + C_2\sin 5t) - (C_3\cos 10t + C_4\sin 10t). \end{aligned}$$

The initial conditions require

$$\frac{1}{10} = C_1 + C_3, \quad 0 = 5C_2 + 10C_4, \quad 0 = 2C_1 - C_3, \quad -2 = 10C_2 - 10C_4.$$

The solution is $C_1 = 1/30$, $C_2 = -2/15$, $C_3 = 1/15$, and $C_4 = 1/15$. Thus,

$$\begin{aligned} x_1(t) &= \frac{1}{30}\cos 5t - \frac{2}{15}\sin 5t + \frac{1}{15}\cos 10t + \frac{1}{15}\sin 10t \text{ m}, \\ x_2(t) &= \frac{1}{15}\cos 5t - \frac{4}{15}\sin 5t - \frac{1}{15}\cos 10t - \frac{1}{15}\sin 10t \text{ m.}\bullet \end{aligned}$$

In the final example, we add a forcing function to one of the masses.

Example 4.23 Repeat Example 4.18, but add a force $4\sin 5t$ acting on M_1.

Solution Displacements $x_1(t)$ and $x_2(t)$ of the masses from their equilibrium positions must satisfy the equations

$$\frac{d^2x_1}{dt^2} = -10x_1 + 4x_2 + 4\sin 5t, \qquad \frac{d^2x_2}{dt^2} = 4x_1 - 4x_2.$$

Initial conditions remain

$$x_1(0) = \frac{1}{10}, \qquad x_1'(0) = 0, \qquad x_2(0) = -\frac{1}{5}, \qquad x_2'(0) = 0.$$

In matrix form,

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -10 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4\sin 5t \\ 0 \end{pmatrix}.$$

Eigenvalues of the matrix are still $\lambda_1 = -2$ and $\lambda_2 = -12$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. If $q_1(t)$ and $q_2(t)$ are components of $\mathbf{x}(t)$ with respect to the eigenvector basis, so that $\mathbf{x}(t) = q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2$, then

$$\frac{d^2}{dt^2}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -12 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where b_1 and b_2 are the components of the vector $(4\sin 5t, 0)$ with respect to the eigenvalue basis. These turn out to be $b_1 = (4/5)\sin 5t$ and $b_2 = -(8/5)\sin 5t$. Equating entries gives the system of differential equations

$$\frac{d^2q_1}{dt^2} = -2q_1 + \frac{4}{5}\sin 5t, \qquad \frac{d^2q_2}{dt^2} = -12q_2 - \frac{8}{5}\sin 5t.$$

Solutions of these are

$$q_1(t) = C_1\cos\sqrt{2}t + C_2\sin\sqrt{2}t - \frac{4}{115}\sin 5t, \qquad q_2(t) = C_3\cos 2\sqrt{3}t + C_4\sin 2\sqrt{3}t + \frac{8}{65}\sin 5t.$$

To evaluate constants, we return to the natural basis

$$\begin{aligned}\mathbf{x}(t) &= q_1(t)\mathbf{v}_1 + q_2(t)\mathbf{v}_2 \\ &= \left(C_1\cos\sqrt{2}t + C_2\sin\sqrt{2}t - \frac{4}{115}\sin 5t\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \left(C_3\cos 2\sqrt{3}t + C_4\sin 10t + \frac{8}{65}\sin 5t\right)\begin{pmatrix} -2 \\ 1 \end{pmatrix}.\end{aligned}$$

When we equate components,

$$\begin{aligned}x_1(t) &= \left(C_1\cos\sqrt{2}t + C_2\sin\sqrt{2}t - \frac{4}{115}\sin 5t\right) - 2\left(C_3\cos 2\sqrt{3}t + C_4\sin 2\sqrt{3}t + \frac{8}{65}\sin 5t\right) \\ &= C_1\cos\sqrt{2}t + C_2\sin\sqrt{2}t - 2C_3\cos 2\sqrt{3}t - 2C_4\sin 2\sqrt{3}t - \frac{84}{299}\sin 5t \\ x_2(t) &= 2\left(C_1\cos\sqrt{2}t + C_2\sin\sqrt{2}t - \frac{4}{115}\sin 5t\right) + \left(C_3\cos 2\sqrt{3}t + C_4\sin 2\sqrt{3}t + \frac{8}{65}\sin 5t\right) \\ &= 2C_1\cos\sqrt{2}t + 2C_2\sin\sqrt{2}t + C_3\cos 2\sqrt{3}t + C_4\sin 2\sqrt{3}t + \frac{16}{299}\sin 5t.\end{aligned}$$

The initial conditions require

$$\frac{1}{10} = C_1 - 2C_3, \quad 0 = \sqrt{2}C_2 - 4\sqrt{3}C_4 - \frac{420}{299}, \quad -\frac{1}{5} = 2C_1 + C_3, \quad 0 = 2\sqrt{2}C_2 + 2\sqrt{3}C_4 + \frac{80}{299}.$$

The solution is $C_1 = -3/50$, $C_2 = 2\sqrt{2}/23$, $C_3 = -2/25$, and $C_4 = -4\sqrt{3}/39$. Thus,

$$x_1(t) = -\frac{3}{50}\cos\sqrt{2}t + \frac{2\sqrt{2}}{23}\sin\sqrt{2}t + \frac{4}{25}\cos 2\sqrt{3}t + \frac{8\sqrt{3}}{39}\sin 2\sqrt{3}t - \frac{84}{299}\sin 5t$$
$$x_2(t) = -\frac{3}{25}\cos\sqrt{2}t + \frac{4\sqrt{2}}{23}\sin\sqrt{2}t - \frac{2}{25}\cos 2\sqrt{3}t - \frac{4\sqrt{3}}{39}\sin 2\sqrt{3}t + \frac{16}{299}\sin 5t.\bullet$$

EXERCISES 4.4

In Exercises 1–2 find positions of the masses in Figure 4.22 given the following values for k_1, k_2, M_1, and M_2, and initial values.

1. $k_1 = 6$, $k_2 = 4$, $M_1 = 1$, $M_2 = 1$; $x_1(0) = 1$, $x_1'(0) = 0$, $x_2(0) = -1$, $x_2'(0) = 0$

2. $k_1 = 6$, $k_2 = 4$, $M_1 = 1$, $M_2 = 1$; $x_1(0) = 0$, $x_1'(0) = 1$, $x_2(0) = 0$, $x_2'(0) = -1$

3. Solve Exercise 2 if damping proportional to velocity with $\beta = 1$ also acts on the masses.

4. Solve Example 4.18 if damping proportional to velocity with $\beta = 8$ also acts on the masses.

5. Solve Exercise 2 if a force $3\sin t$ acts on M_1.

6. Solve Exercise 2 if a force $3\sin\sqrt{2}t$ acts on M_2. Does resonance occur?

In Exercises 7–8 find positions of the masses in Figure 4.24 given the following values for k_1, k_2, k_3, M_1, and M_2, and initial values. Assume that at equilibrium there are no stretches or compressions in the springs.

7. $k_1 = 10$, $k_2 = 20$, $k_3 = 10$, $M_1 = 1/10$, $M_2 = 1/10$; $x_1(0) = 0$, $x_1'(0) = 1$, $x_2(0) = 2$, $x_2'(0) = -1$

8. $k_1 = 200$, $k_2 = 100$, $k_3 = 200$, $M_1 = 1$, $M_2 = 2$; $x_1(0) = 1/10$, $x_1'(0) = -1$, $x_2(0) = -1/10$, $x_2'(0) = 1$

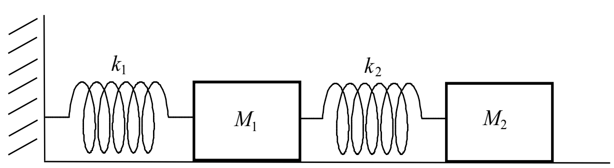
In Exercises 9–10 find positions of the masses in Figure 4.26 given the following values for k_1, k_2, M_1, and M_2, and initial values.

9. $k_1 = 100$, $k_2 = 200$, $M_1 = 2$, $M_2 = 1$; $x_1(0) = 1/200$, $x_1'(0) = 0$, $x_2(0) = -1/100$, $x_2'(0) = 0$

10. $k_1 = 100$, $k_2 = 200$, $M_1 = 2$, $M_2 = 1$; $x_1(0) = 1/40$, $x_1'(0) = -1/2$, $x_2(0) = 0$, $x_2'(0) = 0$

11. Masses M_1 and M_2 are joined by springs with constants k_1 and k_2 as shown below. Verify that when M_2 is pulled a distance s to the right, stretches in the spring are

$$s_1 = \frac{k_2 s}{k_1 + k_2}, \qquad s_2 = \frac{k_1 s}{k_1 + k_2}.$$



12. Show that if the configuration in Exercise 11 is turned vertically with M_2 below M_1, then stretches in the springs at equilibrium are

$$s_1 = \frac{(M_1 + M_2)g}{k_1}, \qquad s_2 = \frac{M_2 g}{k_2}.$$

Answers

1. $x_1(t) = -\frac{1}{5}\cos\sqrt{2}t + \frac{6}{5}\cos 2\sqrt{3}t, \quad x_2(t) = -\frac{2}{5}\cos\sqrt{2}t - \frac{3}{5}\cos 2\sqrt{3}t$

2. $x_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t, \quad x_2(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t$

3. $x_1(t) = -\frac{2}{5\sqrt{7}}e^{-t/2}\sin\frac{\sqrt{7}t}{2} + \frac{12}{5\sqrt{47}}e^{-t/2}\sin\frac{\sqrt{47}t}{2},$

$x_2(t) = -\frac{4}{5\sqrt{7}}e^{-t/2}\sin\frac{\sqrt{7}t}{2} - \frac{6}{5\sqrt{47}}e^{-t/2}\sin\frac{\sqrt{47}t}{2}$

4. $x_1(t) = -\frac{3(7+2\sqrt{14})}{700}e^{(-4+\sqrt{14})t} + \frac{3(-7+2\sqrt{14})}{700}e^{-(4+\sqrt{14})t} + \frac{6}{25}e^{-2t} - \frac{2}{25}e^{-6t}$

$x_2(t) = -\frac{3(7+2\sqrt{14})}{350}e^{(-4+\sqrt{14})t} + \frac{3(-7+2\sqrt{14})}{350}e^{-(4+\sqrt{14})t} - \frac{3}{25}e^{-2t} + \frac{1}{25}e^{-6t}$

5. $x_1(t) = -\frac{2\sqrt{2}}{5}\sin\sqrt{2}t + \frac{9\sqrt{3}}{55}\sin 2\sqrt{3}t + \frac{9}{11}\sin t, \quad x_2(t) = -\frac{4\sqrt{2}}{5}\sin\sqrt{2}t - \frac{9\sqrt{3}}{110}\sin 2\sqrt{3}t + \frac{12}{11}\sin t$

6. $x_1(t) = -\left(\frac{2\sqrt{2}}{5} + \frac{3}{20}\right)\sin\sqrt{2}t + \frac{9\sqrt{3}}{55}\sin 2\sqrt{3}t - \frac{3\sqrt{2}t}{10}\cos\sqrt{2}t$

$x_2(t) = -\frac{4\sqrt{2}}{5}\sin\sqrt{2}t - \frac{9\sqrt{3}}{110}\sin 2\sqrt{3}t - \frac{3\sqrt{2}t}{5}\cos\sqrt{2}t,$ Yes

7. $x_1(t) = \cos 10t - \cos 10\sqrt{5}t + \frac{\sqrt{5}}{50}\sin 10\sqrt{5}t, \quad x_2(t) = \cos 10t + \cos 10\sqrt{5}t - \frac{\sqrt{5}}{10}\sin 10\sqrt{5}t$

8. $x_1(t) = \frac{\sqrt{17}}{340}\Big\{(7+\sqrt{17})\cos 5\sqrt{9+\sqrt{17}}t - \left(\frac{14+2\sqrt{17}}{\sqrt{9+\sqrt{17}}}\right)\sin 5\sqrt{9+\sqrt{17}}t$

$+(\sqrt{17}-7)\cos 5\sqrt{9-\sqrt{17}}t + \left(\frac{14-2\sqrt{17}}{\sqrt{9-\sqrt{17}}}\right)\sin 5\sqrt{9-\sqrt{17}}t\Big\},$

$x_2(t) = \frac{\sqrt{17}}{340}\Big\{(1-\sqrt{17})\cos 5\sqrt{9+\sqrt{17}}t + \left(\frac{2\sqrt{17}-2}{\sqrt{9+\sqrt{17}}}\right)\sin 5\sqrt{9+\sqrt{17}}t$

$-(1+\sqrt{17})\cos 5\sqrt{9-\sqrt{17}}t + \left(\frac{2+2\sqrt{17}}{\sqrt{9-\sqrt{17}}}\right)\sin 5\sqrt{9-\sqrt{17}}t\Big\}$

9. $x_1(t) = \frac{1}{13\,200}\left[(33-7\sqrt{33})\cos 5\sqrt{7-\sqrt{33}}t + (33+7\sqrt{33})\cos 5\sqrt{7+\sqrt{33}}t\right]$

$x_2(t) = \frac{1}{6600}\left[(5\sqrt{33}-33)\cos 5\sqrt{7-\sqrt{33}}t - (5\sqrt{33}+33)\cos 5\sqrt{7+\sqrt{33}}t\right]$

10. $x_1(t) = \frac{1}{2640}\big[(33+\sqrt{33})(\cos 5\sqrt{7-\sqrt{33}}t - \sqrt{7+\sqrt{33}}\sin 5\sqrt{7-\sqrt{33}}t)$

$+ (33-\sqrt{33})(\cos 5\sqrt{7+\sqrt{33}}t - \sqrt{7-\sqrt{33}}\sin 5\sqrt{7+\sqrt{33}}t)\big],$

$x_2(t) = \frac{\sqrt{33}}{330}\Big[\cos 5\sqrt{7-\sqrt{33}}t - \frac{4}{\sqrt{7-\sqrt{33}}}\sin 5\sqrt{7-\sqrt{33}}t$

$- \cos 5\sqrt{7+\sqrt{33}}t + \frac{4}{\sqrt{7+\sqrt{33}}}\sin 5\sqrt{7+\sqrt{33}}t\Big]$