Assignment 2

1. (a) Show that the linear operator L on $\mathcal{G}^3$ that reflects vectors in the plane $Ax+By+Cz=0$ transforms vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ according to

$$L(\mathbf{v}) = \frac{1}{A^2+B^2+C^2}\langle (B^2+C^2-A^2)v_x - 2ABv_y - 2ACv_z, -2ABv_x + (A^2+C^2-B^2)v_y - 2BCv_z, \\ -2ACv_x - 2BCv_y + (A^2+B^2-C^2)v_z \rangle.$$

(b) Find the matrix of the linear operator with respect to the natural basis of $\mathcal{G}^3$.

(c) Find a basis for the null space of the operator.

2. The first five Legendre polynomials are

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \frac{1}{2}(3x^2-1), \quad p_3(x) = \frac{1}{2}(5x^3-3x), \quad p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Find the matrix of the linear transformation L from $P_4(x)$ to $P_3(x)$ that maps functions $p(x)$ in $P_4(x)$ according to

$$L\big(p(x)\big) = \frac{d^2p}{dx^2} - 2\frac{dp}{dx}.$$

3. (a) Find a basis for the kernel of the linear transformation L that maps polynomials $p(x)$ in $P_2(x)$ to $\mathcal{R}$ according to

$$L\big(p(x)\big) = \int_a^b p(x)\,dx.$$

(b) What is the range of the transformation?

(c) Is the transformation into, onto, and/or 1-1? Justify your answsers.

4. (a) Verify that the transformation L that maps $M_{2,2}(\mathcal{R})$ to $P_3(x)$ according to

$$L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ax^3 + (a-b)x^2 + (a-c)x + d,$$

is linear.

(b) Find the matrix of the linear transformation with respect to natural bases in both spaces.

(c) Find the kernel and range of the transformation?

(d) Show that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and $\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ are bases for $M_{2,2}(\mathcal{R})$ and $P_3(x)$.

(e) Find the matrix of the linear transformation with respect to these bases.