3132 Midterm 2 2024 Solutions

20 **1.** If a Frobenius solution $y(x)=\sum_{n=0}^{\infty} a_n x^{n+r}$ is assumed for the differential equation

$$2x^2\frac{d^2y}{dx^2}+x(1-x)\frac{dy}{dx}-y=0,$$

find the indicial roots and the recurrence relation for the a_n corresponding to the larger indicial root, simplified as much as possible. Do **NOT** iterate the recurrence relation.

When we substitute $y(x)=\sum_{n=0}^{\infty} a_n x^{n+r}$ into the differential equation

$$\begin{aligned}0&=\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+\sum_{n=0}^{\infty}-(n+r)a_nx^{n+r+1}+\sum_{n=0}^{\infty}-a_nx^{n+r}\\&=\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+\sum_{n=1}^{\infty}-(n+r-1)a_{n-1}x^{n+r}+\sum_{n=0}^{\infty}-a_nx^{n+r}\\&=a_0[2r(r-1)+r-1]x^r+\sum_{n=1}^{\infty}[2(n+r)(n+r-1)a_n+(n+r)a_n-(n+r-1)a_{n-1}-a_n]x^{n+r}.\end{aligned}$$

The indicial equation is

$$0=2r(r-1)+r-1=2r^2-r-1=(2r+1)(r-1)\quad\Longrightarrow\quad r=-1/2,1.$$

When $r=1$, the recurrence relation is given by

$$\begin{aligned}2(n+1)(n)a_n+(n+1)a_n-na_{n-1}-a_n&=0\\ [2(n+1)n+(n+1)-1]a_n&=na_{n-1}\\ a_n&=\frac{na_{n-1}}{2n^2+3n}=\frac{a_{n-1}}{2n+3},\quad n\geq 1\end{aligned}$$

5 **2.** Find value(s) of c, if any, for which the functions x and $x-x^3$ are orthogonal on the interval $0\leq x\leq c$ with respect to the weight function $w(x)=x$.

For orthogonality, we require

$$0=\int_0^c x(x-x^3)x\,dx=\left\{\frac{x^4}{4}-\frac{x^6}{6}\right\}_0^c=\frac{c^4}{4}-\frac{c^6}{6}=\frac{c^4}{12}(3-2c^2).$$

Hence, $3-2c^2=0\quad\Longrightarrow\quad c=\pm\sqrt{3/2}$, only $c=\sqrt{3/2}$ being acceptable.

5 **3.** Find all singular points for the differential equation

$$\sin(x-1)\frac{d^y}{dx^2} + (x-1)\frac{dy}{dx} + 3y = 0.$$

Since $\dfrac{3}{\sin(x-1)}$ does not have a convergent Taylor series when $\sin(x-1) = 0 \implies x-1 = n\pi$, where n is an integer, singular points are $x = 1 + n\pi$.

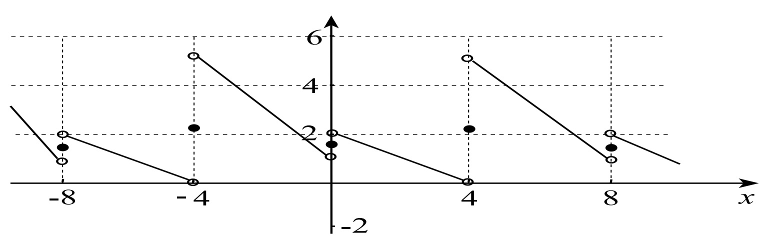
20 **4.** The function

$$f(x) = \begin{cases} 1-x, & -4 < x < 0, \\ -(x-4)/2, & 0 < x < 4, \end{cases} \qquad f(x+8) = f(x),$$

is to be represented in Fourier series form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

(a) What is the value for L?
(b) Use integration to find a_0. Explain why this is to be expected.
(c) Set up, but do **NOT** evaluate, definite integral(s) for coefficients b_n. Your integrand(s) must contain the function(s) $1-x$ and/or $-(x-4)/2$.
(d) On the grid below, draw a graph of the function to which the Fourier series converges on the interval $-10 \le x \le 10$.



(a) $L = 4$
(b)

$$a_0 = \frac{1}{4}\int_{-4}^{4} f(x)\,dx = \frac{1}{4}\int_{-4}^{0}(1-x)\,dx + \frac{1}{4}\int_{0}^{4}(1/2)(4-x)\,dx$$

$$= \frac{1}{4}\left\{x - \frac{x^2}{2}\right\}_{-4}^{0} + \frac{1}{8}\left\{4x - \frac{x^2}{2}\right\}_{0}^{4} = 4$$

We know that $a_0/2 = 2$ should be the average value of $f(x)$ over one full period. Since the average value on the interval $-4 \le x \le 0$ is 3, and the average value of the interval $0 \le x \le 4$ is 1, the average value over $-4 \le x \le 4$ is 2.

(c) $b_n = \dfrac{1}{4}\displaystyle\int_{-4}^{4} f(x)\sin\frac{n\pi x}{4}dx = \frac{1}{4}\int_{-4}^{0}(1-x)\sin\frac{n\pi x}{4}dx + \frac{1}{4}\int_{0}^{4}\frac{1}{2}(x-4)\sin\frac{n\pi x}{4}dx$