MATH 3132 Tutorial 10

1. Find eigenvalues and eigenfunctions of the Sturm-Liouville system

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad 1 < x < 10, \qquad y(1) = 0, \quad y(10) = 0,$$

(a) directly, and (b) making the substitution $z = x - 1$.

2. Find eigenvalues, eigenfunctions and the weight function for the Sturm-Liouville system

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + \lambda y = 0, \quad 0 < x < 1, \qquad y(0) = 0, \quad y(1) = 0.$$

3. Repeat Question 2 for the system

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0, \quad 1 < x < e, \qquad y(1) = 0, \quad y(e) = 0$$

Hint: Set $y = x^m$. and find values for m.

4 The Sturm-Liouville system

$$\begin{aligned} \frac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 2L, \\ y(0) = y(2L), \\ y'(0) = y'(2L), \end{aligned}$$

is said to be a periodic system. Find its eigenvalues and eigenfunctions. Recognize them?

5. Expand the function $f(x) = 3\cos\frac{5\pi x}{8} - 2\cos\frac{9\pi x}{8}$, $0 < x < 4$, in terms of the eigenfunctions of the Sturm-Liouville system

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 4, \qquad y'(0) = 0 = y(4).$$

6. (a) Expand the fuction $f(x) = L - x$, $0 < x < L$, in terms of the eigenfunctions of the Sturm-Liouville system

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < L, \qquad y'(0) = 0 = y(L).$$

(b) To what does the expansion converge at $x = 0$ and $x = L$? Hint:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}, \qquad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

7. Eigenfunctions $y_n(x)$ of Sturm-Liouville systems are said to be normalized when their norms are equal to 1, where the norm, denoted by $\|y_n(x)\|$, is defined by

$$\|y_n(x)\|^2 = \int_a^b p(x)[y_n(x)]^2\,dx.$$

Find normalized eigenfunctions for the following Sturm-Liouville systems:

(a) $y'' + \lambda y = 0, \quad 0 < x < L, \qquad y(0) = 0 = y(L).$

(b) $y'' + \lambda y = 0, \quad 0 < x < L, \qquad y'(0) = 0 = y(L).$

(c) $y'' + 2y' + \lambda y = 0, \quad 0 < x < L, \qquad y'(0 = 0 = y'(L).$

Answers:

1. $n^2\pi^2/81, \quad \sin\dfrac{n\pi(x-1)}{9}$

2. $n^2\pi^2 + 1/4, \quad e^{x/2}\sin n\pi x, \quad e^{-x}$

3. $n^2\pi^2, \quad \sin(n\pi \ln x), \quad 1/x$

4. $n^2\pi^2/L^2$, $n \geq 0$: $\quad y_0(x) = A$, $y_n(x) = A\cos\dfrac{n\pi x}{L} + B\sin\dfrac{n\pi x}{L}$

5. $3\cos\dfrac{5\pi x}{8} - 2\cos\dfrac{9\pi x}{8}$

6.(a) $\dfrac{4L}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\left[\frac{\pi}{2n-1} + \frac{2(-1)^n}{(2n-1)^2}\right]\sin\frac{(2n-1)\pi x}{2L}$ (b) 0,0

7.(a) $\sqrt{\dfrac{2}{L}}\sin\dfrac{n\pi x}{L}$ (b) $\sqrt{\dfrac{2}{L}}\cos\dfrac{(2n-1)\pi x}{2L}$

(c) $y_0(x) = \sqrt{\dfrac{2}{e^{2L}-1}}, \quad y_n(x) = \dfrac{\sqrt{2L}}{\sqrt{L^2+n^2\pi^2}}e^{-x}\left(\dfrac{n\pi}{L}\cos\dfrac{n\pi x}{L} + \sin\dfrac{n\pi x}{L}\right)$