MATH 3132 Tutorial 11

1. Show that the Laplacian of a function $V(r,\theta)$ in polar coordinates is

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

In Problems 2–5, describe Dirichlet and Neumann bounday conditions on the boundary of the region for the partial differential equation.

2. $\dfrac{\partial U}{\partial t} = k\dfrac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \quad t > 0$

3. $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \quad t > 0$

4 $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} = 0, \qquad 0 < r < a, \quad -\pi < \theta \le \pi$

5 $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} = 0, \qquad 0 < r < a, \quad 0 < \theta < \pi$

In Problems 6–8, set up, but do NOT solve, an initial bounday value problem for the required temperature. Assume that the medium is isotropic and homogeneous.

6. A cylindrical rod has flat ends at $x = 0$ and $x = L$, and insulated sides. At time $t = 0$, the temperature in the rod is a function $f(x)$, $0 \le x \le L$, of x only. For $t > 0$ end $x = L$ is held at constant temperature U_L, and end $x = 0$ is insulated.

7. Repeat Problem 6 except that the temperature at end $x = L$ is changed from 0° C to 100° C at a constant rate over a period of T seconds, and maintained at 100° C thereafter.

8. Repeat Problem 6 except that heat is added to the end $x = 0$ at a constant rate $Q_0 > 0$ W/m^2 uniformly over the end, and is removed at a variable rate $Q_L(t) > 0$ W at $x = L$ uniformly over the end. Heat is generated at the rate $g(x,t)$ per unit volume per unit time.

In problesm 9–10, set up, but do NOT solve, an initial boundary value problem for displacements in a string. Assume that density and tension in the string are constant.

9. A taut string is stretched between $x = 0$ and $x = L$ along the x-axis. End $x = 0$ is fixed on the x-axis, but end $x = L$ is free to slide frictionlessly along a vertical support. It is given an initial displacement $f(x)$ and initial velocity $g(x)$.

10. The ends of a taut string are fixed at $x = 0$ and $x = L$ on the x-axis. The string is initially at rest along the x-axis, and is then allowed to drop under its own weight. The static deflection of the string is the shape of the curve when it hangs motionless under gravity. Find this shape.

Answers:

2. Dirichlet: $U(0,t) = f_1(t)$, $U(L,t) = f_2(t)$ Neumann: $U_x(0,t) = g_1(t)$, $U_x(L,t) = g_2(t)$

3. Dirichlet: $y(0,t) = f_1(t)$, $y(L,t) = f_2(t)$ Neumann: $y_x(0,t) = g_1(t)$, $y_x(L,t) = g_2(t)$

4. Dirichlet: $V(a,\theta) = f(\theta)$ Neumann: $V_r(a,\theta) = g(\theta)$

5. Dirichlet: $V(a,\theta) = f_1(\theta)$, $V(r,0) = f_2(r)$, $V(r,\pi) = f_3(r)$
Neumann: $V_r(a,\theta) = g_1(r)$, $V_\theta(r,0) = g_2(r)$, $V_\theta(r,\pi) = g_3(r)$

6.

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \quad t > 0,\\
U_x(0,t) &= 0, \quad t > 0,\\
U(L,t) &= U_L, \quad t > 0,\\
U(x,0) &= f(x), \quad 0 < x < L.
\end{aligned}$$

7.

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \quad t > 0,\\
U_x(0,t) &= 0, \quad t > 0,\\
U(L,t) &= 100t/T, \text{ for } 0 < t \le T, \text{ and } T, \text{ for } t > T,\\
U(x,0) &= f(x), \quad 0 < x < L.
\end{aligned}$$

8.

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \qquad 0 < x < L, \quad t > 0,\\
U_x(0,t) &= -Q_0/\kappa, \quad t > 0,\\
U_x(L,t) &= -Q_L(t)/(\kappa A), \quad t > 0, (\quad (A \text{ is the area of the end of the rod})\\
U(x,0) &= f(x), \quad 0 < x < L.
\end{aligned}$$

9.

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \quad t > 0,\\
y(0,t) &= 0, \quad t > 0,\\
y_x(L,t) &= 0, \quad t > 0,\\
y(x,0) &= f(x), \quad 0 < x < L,\\
y_t(x,0) &= g(x), \quad 0 < x < L.
\end{aligned}$$

10.

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2} - g, \qquad 0 < x < L, \quad t > 0,\\
y(0,t) &= 0, \quad t > 0,\\
y(L,t) &= 0, \quad t > 0,\\
y(x,0) &= 0, \quad 0 < x < L,\\
y_t(x,0) &= 0, \quad 0 < x < L.
\end{aligned}$$

Static deflections: $y(x) = \dfrac{gx(x-L)}{2c^2}$