Solution to Exam Fall 2024

10 **1.** Evaluate the line integral

$$\oint_C \frac{y}{x^2+2y^2}dx + \frac{x}{x^2+2y^2}dy$$

where C is the curve $x^2+2y^2=4$.

Solution 1: If we set $x = 2\cos t$, $y = \sqrt{2}\sin t$, $0 \le t \le 2\pi$, then

$$\begin{aligned}
\oint_C \frac{y}{x^2+2y^2}dx + \frac{x}{x^2+2y^2}dy &= \int_0^{2\pi} \left[\frac{\sqrt{2}\sin t}{4}(-2\sin t\,dt) + \frac{2\cos t}{4}(\sqrt{2}\cos t\,dt)\right] \\
&= \frac{1}{\sqrt{2}}\int_0^{2\pi} (\cos^2 t - \sin^2 t)\,dt = \frac{1}{\sqrt{2}}\int_0^{2\pi} \cos 2t\,dt \\
&= \frac{1}{\sqrt{2}}\left\{\frac{1}{2}\sin 2t\right\}_0^{2\pi} = 0.
\end{aligned}$$

Solution 2: Since $x^2+2y^2=4$ on the curve, we can rewrite the line integral as

$$\oint_C \frac{y}{4}dx + \frac{x}{4}dy = \frac{1}{4}\oint_C (y\,dx + x\,dy).$$

Since $\nabla(xy) = y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$, this line integral is independent of path in the entire xy-plane. Because the curve is closed, its value is zero.

Solution 3: Since $x^2+2y^2=4$ on the curve, we can rewrite the line integral as

$$\oint_C \frac{y}{4}dx + \frac{x}{4}dy = \frac{1}{4}\oint_C (y\,dx + x\,dy).$$

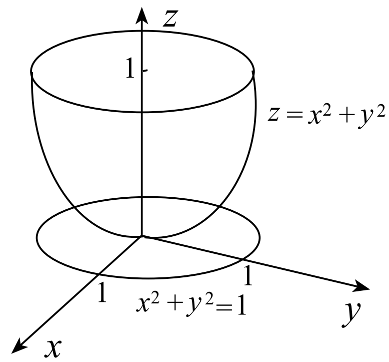
We can now use Green's Theorem to replace the line integral with a double integral over the interior R of C,

$$\oint_C \frac{y}{4}dx + \frac{x}{4}dy = \frac{1}{4}\iint_R (0)\,dA = 0.$$

15 **2.** Evaluate the surface integral

$$\iint_S (x\,\hat{\mathbf{i}} - y^2 z\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\,dS$$

were S is that part of the surface $z = x^2 + y^2$ below the plane $z = 1$, and $\hat{\mathbf{n}}$ is the downward pointing unit normal to S.



Solution 1: A normal vector to S is $\nabla(x^2 + y^2 - z) = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - \hat{\mathbf{k}}$ so that $\hat{\mathbf{n}} = \dfrac{(2x, 2y, -1)}{\sqrt{1 + 4x^2 + 4y^2}}$.
If we denote the integral by I, then,

$$I = \iint_{S_{xy}} \frac{(2x^2 - 2y^3 z - xz^2)}{\sqrt{1 + 4x^2 + 4y^2}} \sqrt{1 + (2x)^2 + (2y)^2}\,dA = \iint_{S_{xy}} [2x^2 - 2y^3(x^2 + y^2) - x(x^2 + y^2)^2]\,dA,$$

where S_{xy} is the interior of the circle $x^2 + y^2 = 1$ in the xy-plane. Since $2y^3(x^2 + y^2)$ is an odd function of y and S_{xy} is symmetric about the x-axis, this term integrates to zero. Similarly, $x(x^2 + y^2)^2$ is an odd function of x, and S_{xy} is symmetric about the y-axis so this term also integrates to zero. On the remaining term, we use polar coordinates,

$$\begin{aligned} I &= 8 \int_0^{\pi/2} \int_0^1 r^2 \cos^2\theta\, r\,dr\,d\theta = 8 \int_0^{\pi/2} \left\{ \frac{r^4}{4} \cos^2\theta \right\}_0^1 d\theta \\ &= 2 \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta = \left\{ \theta + \frac{1}{2} \sin 2\theta \right\}_0^{\pi/2} = \frac{\pi}{2}. \end{aligned}$$

Solution 2: We close the surface with that part of the plane $z = 1$ inside $x^2 + y^2 = 1$, call it S'. We use the divergence theorem on

$$I' = \oiint_{S+S'} (x\,\hat{\mathbf{i}} - y^2 z\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\,dS = \iiint_V (1 - 2yz + 2xz)\,dV,$$

where V is the volume bounded by S and S'. Since $2yz$ is an odd function of y and V is symmetric about the xz-plane, this term integrates to zero. Similarly, because $2xz$ is odd in x and V is symmetrc about the yz-plane, this term also integrates to zero. With cylindrical coordinates,

$$I' = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^1 r\,dz\,dr\,d\theta = 4 \int_0^{\pi/2} \int_0^1 r(1 - r^2)\,dr\,d\theta = 4 \int_0^{\pi/2} \left\{ \frac{r^2}{2} - \frac{r^4}{4} \right\}_0^1 d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}.$$

Since

$$\iint_{S'} (x\,\hat{\mathbf{i}} - y^2 z\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\,dS = \iint_{S'} (x\,\hat{\mathbf{i}} - y^2 z\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{k}}\,dS = \iint_{S'_{xy}} xz^2\,dA = \iint_{S'_{xy}} x\,dA = 0,$$

it follows that

$$I = I' - \iint_{S'} (x\,\hat{\mathbf{i}} - y^2 z\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\,dS = \frac{\pi}{2}.$$

10 **3.** Find the Fourier series of the function

$$f(x) = \begin{cases} -x^2, & -1 < x < 0 \\ x^2, & 0 < x < 1 \end{cases} \qquad\qquad f(x+2) = f(x).$$

Draw a graph of the function to which the Fourier series converges on the interval $-4 \le x \le 4$.

Since the function is odd, we find the Fourier sine series,

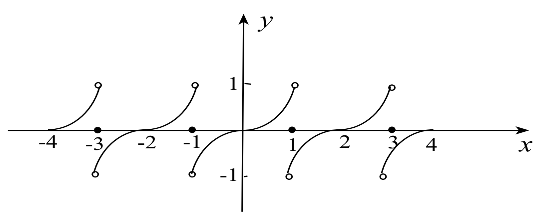
$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} b_n \sin n\pi x,$$

where

$$\begin{aligned} b_n &= 2\int_0^1 x^2 \sin n\pi x\, dx = 2\left[\left\{-\frac{x^2}{n\pi}\cos n\pi x\right\}_0^1 - \int_0^1 -\frac{2x}{n\pi}\cos n\pi x\, dx\right] \\ &= 2\left[\frac{(-1)^{n+1}}{n\pi} + \frac{2}{n\pi}\left\{\frac{x}{n\pi}\sin n\pi x\right\}_0^1 - \frac{2}{n\pi}\int_0^1 \frac{\sin n\pi x}{n\pi}\, dx\right] \\ &= \frac{2(-1)^{n+1}}{n\pi} - \frac{4}{n^2\pi^2}\left\{-\frac{\cos n\pi x}{n\pi}\right\}_0^1 = \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}[(-1)^n - 1]. \end{aligned}$$

Hence,

$$\frac{f(x+) + f(x-)}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^2}[(-1)^n - 1]\right]\sin n\pi x,$$



20 **4.** (a) Show that the indicial roots of a Frobenius solution $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ of the differential equation

$$(x - x^2)\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

are $r = 0, 4$.

(b) Assuming that the recurrence relation for the a_n corresponding to $r = 0$ is

$$(n-3)a_{n+1} = (n-2)a_n, \qquad n \geq 0,$$

find the solution to the differential equation. Is it a general solution? Explain.
Five bonus marks if you can express the solution in closed form (that is, no series).

(a) When we substitute the Frobenius solution into the differential equation

$$\begin{aligned}0 &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} -(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} -3(n+r)a_n x^{n+r-1} \\ &\quad + \sum_{n=0}^{\infty} 2a_n x^{n+r} \\ &= \sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} + \sum_{n=0}^{\infty} -(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=-1}^{\infty} -3(n+r+1)a_{n+1}x^{n+r} \\ &\quad + \sum_{n=0}^{\infty} 2a_n x^{n+r}.\end{aligned}$$

The indicial equation comes from setting the coefficient of the lowest power of x equal to zero,

$$0 = r(r-1)a_0 - 3ra_0 = r(r-4)a_0.$$

Thus, indicial roots are $r = 0, 4$.
(b) When $n = 0$, $-3a_1 = -2a_0 \implies a_1 = \frac{2a_0}{3}$
When $n = 1$, $-2a_2 = -a_1 \implies a_2 = \frac{a_0}{3}$
When $n = 2$, $a_3 = 0$.
For $n > 3$, we write $a_{n+1} = \dfrac{n-2}{n-3}a_n$.
When $n = 4$, $a_5 = 2a_4$.
When $n = 5$, $a_6 = \frac{3}{2}a_5 = 3a_4$.
When $n = 6$, $a_7 = \frac{4}{3}a_3 = 4a_4$.
The solution is

$$\begin{aligned}y(x) &= a_0\left(1 + \frac{2x}{3} + \frac{x^2}{3}\right) + a_4\left(x^4 + 2x^5 + 3x^6 + \cdots\right) \\ &= a_0\left(1 + \frac{2x}{3} + \frac{x^2}{3}\right) + a_4\sum_{n=4}^{\infty}(n-3)x^n.\end{aligned}$$

This is a general solution since it contains two arbitrary constants.

5 **5.** Find value(s) of constant k if the functions $f(x) = x^2 + kx + 1$ and $g(x) = x$ are to be orthogonal on the interval $0 \le x \le 2$ if the weight function is $w(x) = x$.

Fo orthogonality,

$$0 = \int_0^2 (x^2 + kx + 1)(x)(x)\,dx = \left\{ \frac{x^5}{5} + \frac{kx^4}{4} + \frac{x^3}{3} \right\}_0^2 = \frac{32}{5} + 4k + \frac{8}{3}.$$

This implies that $k = -34/15$

6 **6.** Set up, but do **NOT** solve, an initial boundary value problem for displacement $y(x,t)$ of a taut string of length L with constant tension τ, and constant mass per unit length ρ. The string is given initial displacement $f(x)$ and initial velocity $g(x)$. Take gravity and a damping force proportional to velocity into account. The right end of the string is fixed on the x-axis, and the left end is looped around the y-axis and is free to move vertically along the y-axis. Identify any additional constants that you introduce into the problem.

The initial boundary value problem is

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= \frac{\tau}{\rho}\frac{\partial^2 y}{\partial x^2} - 9.81 - \beta\frac{\partial y}{\partial t}, \quad 0 < x < L, \quad t > 0,\\
y_x(0,t) &= 0, \quad t > 0,\\
y(L,t) &= 0, \quad t > 0,\\
y(x,0) &= f(x), \quad 0 < x < L,\\
y_t(x,0) &= g(x), \quad 0 < x < L.
\end{aligned}$$

β is a damping constant.

22 **7.** Solve the following boundary-value problem

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= 0, \quad 0 < x < L, \quad 0 < y < L,\\
V(x,0) &= 0, \quad 0 < x < L,\\
V(x,L) &= 0, \quad 0 < x < L,\\
V(0,y) &= 1, \quad 0 < y < L,\\
V(L,y) &= 0, \quad 0 < y < L.
\end{aligned}$$

Justify each step in your solution.

We begin by finding separated functions $V(x,y) = X(x)Y(y)$ satisfying the PDE and the first, second and fourth boundary conditions. The PDE requires

$$X''Y + XY'' = 0 \quad\Longrightarrow\quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \quad\Longrightarrow\quad X'' - \lambda X = 0, \quad Y'' + \lambda Y = 0.$$

The first, second and fourth boundary conditions require $X(L) = 0$, $Y(0) = 0$, and $Y(L) = 0$. Eigenvalues of the Sturm-Liouville system in $Y(y)$ are $\lambda_n = n^2\pi^2/L^2$ with corresponding eigenfunctions $Y_n(y) = \sin\frac{n\pi y}{L}$. Solutions of $X'' - (n^2\pi^2/L^2)X = 0$ are $X_n(x) = C_1 e^{n\pi x/L} + C_2 e^{-n\pi x/L}$. The condition $X(L) = 0$ requires $0 = C_1 e^{n\pi} + C_2 e^{-n\pi} \implies C_2 = -C_1 e^{2n\pi}$. Thus, separated functions are

$$\begin{aligned}
X_n(x)Y_n(y) &= C_n\left(e^{n\pi x/L} - e^{2n\pi}e^{-n\pi x/L}\right)\sin\frac{n\pi y}{L} = C_n e^{n\pi}\left[e^{-n\pi(1-x/L)} - e^{n\pi(1-x/L)}\right]\sin\frac{n\pi y}{L}\\
&= B_n\left[e^{-n\pi(1-x/L)} - e^{n\pi(1-x/L)}\right]\sin\frac{n\pi y}{L}.
\end{aligned}$$

Because the PDE and the first, second and fourth boundary conditions are linear and homogeneous, we superpose separated functions and take

$$V(x,y) = \sum_{n=1}^{\infty} B_n\left[e^{-n\pi(1-x/L)} - e^{n\pi(1-x/L)}\right]\sin\frac{n\pi y}{L}.$$

The nonhomogeneous boundary condition requires

$$1 = \sum_{n=1}^{\infty} B_n(e^{-n\pi} - e^{n\pi})\sin\frac{n\pi y}{L}.$$

This implies that $B_n(e^{-n\pi} - e^{n\pi})$ are coefficients in the eigenfunction expansion of the function $f(y) = 1$; that is,

$$B_n(e^{-n\pi} - e^{n\pi}) = 2\int_0^1 \sin\frac{n\pi y}{L}\,dy = \frac{2}{n\pi}[1 + (-1)^{n+1}].$$

Thus,

$$\begin{aligned}
V(x,y) &= \sum_{n=1}^{\infty} \frac{2[1 + (-1)^{n+1}]}{n\pi(e^{-n\pi} - e^{n\pi})}\left[e^{-n\pi(1-x/L)} - e^{n\pi(1-x/L)}\right]\sin\frac{n\pi y}{L}\\
&= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)[e^{-(2n-1)\pi} - e^{(2n-1)\pi}]}\left[e^{-(2n-1)\pi(1-x/L)} - e^{(2n-1)\pi(1-x/L)}\right]\sin\frac{(2n-1)\pi y}{L}.
\end{aligned}$$

12 **8.** The nonhomogeneous initial, boundary-value problem

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} + x, \quad 0 < x < L, \quad t > 0,\\
U(0,t) &= U_0, \quad t > 0, \qquad (U_0 \text{ a constant})\\
U_x(L,t) &= Q, \quad t > 0, \qquad (Q \text{ a constant})\\
U(x,0) &= f(x), \quad 0 < x < L,
\end{aligned}$$

can be solved by splitting $U(x,t)$ into two parts

$$U(x,t) = z(x,t) + \psi(x).$$

Find $\psi(x)$ and the initial boundary value problem for $z(x,t)$. Do **NOT** attempt to find $z(x,t)$.

$\psi(x)$ must satisfy

$$0 = k\frac{d^2\psi}{dx^2} + x, \quad \psi(0) = U_0, \quad \psi'(L) = Q.$$

A general solution of the differential equation is $\psi(x) = -\dfrac{x^3}{6k} + Ax + B$. The boundary conditions require

$$B = U_0, \quad Q = -\frac{L^2}{2k} + A.$$

Thus, $\psi(x) = -\dfrac{x^3}{6k} + \left(Q + \dfrac{L^2}{2k}\right)x + U_0$. The initial boundary problem for $z(x,t)$ is

$$\begin{aligned}
\frac{\partial z}{\partial t} &= k\frac{\partial^2 z}{\partial x^2}, \quad 0 < x < L, \quad t > 0,\\
z(0,t) &= 0, \quad t > 0,\\
z_x(L,t) &= 0, \quad t > 0,\\
z(x,0) &= f(x) - \psi(x), \quad 0 < x < L.
\end{aligned}$$