Sample Exam

1. Evaluate the line integral $\oint_C (2xy^3 - xy)\,dx + (3x^2y^2 + xy^2)\,dy$ where C traverses once counter-clockwise around the triangle in the xy-plane that bounds the area enclosed by the curves $y = -x$, y=0, and $x = -1$.

2. Evaluate the surface integral

$$\iint_S [x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - (1 + xz)\,\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\,dS$$

where S is that part of the surface $z = x^2 + y^2 - 4$ that lies below the xy-plane and $\hat{\mathbf{n}}$ is the unit upper normal to S.

3. Evaluate the surface integral

$$\oiint_S (x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} - xy^3\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\,dS$$

where S is the closed surface enclosing the volume in the first octant bounded by the surfaces $x + y + z = 1$, $x = 0$, $y = 0$, and $z = 0$, and $\hat{\mathbf{n}}$ is the unit inward pointing normal to S.

4. Consider the differential equation

$$(2x^2 + 1)y'' - 4y = 0.$$

A solution of the above differential equation can be represented by its Maclaurin series $y = \sum_{n=0}^{\infty} a_n x^n$.

(a) Find a **recurrence relation** for the a_n and simplify it as much as possible. Do **NOT** attempt to iterate the recurrence relation; that is, do **NOT** solve for the a_n.

(b) Using only the properties of the differential equation, determine the open interval of convergence for any Maclaurin series solution. Explain your reasoning.

5. When a solution of the differential equation $(x^2 - 3)y'' - 6y = 0$ is represented in terms of its Maclaurin series $y = \sum_{n=0}^{\infty} c_n x^n$, its coefficients a_n satisfy the recurrence relation

$$a_{n+2} = \frac{n-3}{3(n+1)} a_n, \quad n \geq 0.$$

(a) Use the recurrence relation to find a general solution of the differential equation. Write any infinite series using sigma notation and simplify as much as possible.

(b) Determine a guaranteed open interval of convergence for each of your two linearly independent solutions. Explain your reasoning.

6. (a) Calculate the Fourier sine series of $f(x)$ on the interval $0 \leq x \leq 3$, where $f(x) = 3 + x$, $0 \leq x \leq 3$, and simplify your answer.

(b) Draw a graph of the function to which the Fourier series in part (a) converges for $-9 \leq x \leq 9$, and indicate clearly the scales on the axes, and the value at each point of discontinuity.

7. A thin laterally insulated rod has diffusivity 7 and length 4. The equation for heat conduction in this rod is

$$\frac{\partial U}{\partial t} = 7\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < 4, \quad t > 0,$$

where $U(x,t)$ is the temperature at the point x and time t. The boundary conditions are

$$\frac{\partial U(0,t)}{\partial x} = \frac{\partial U(4,t)}{\partial x} = 0, \quad t > 0.$$

Initially (at time $t = 0$), the temperature of the rod is given by

$$U(x,0) = 10 - x, \quad 0 < x < 4.$$

(a) Solve the problem for $U(x,t)$ by using separation of variables.
(b) What is the physical interpretation of the boundary conditions?
(c) Find the limit of $U(x,t)$ as $t \to \infty$ (if possible), and give a brief physical explanation of your answer.

8. A vibrating string has length 10 metres and at equilibrium it lies along the x-axis from $x = 0$ to $x = 10$. Let $y(x,t)$ be the displacement from the x-axis. The left end is attached to the x-axis. The right end can slide vertically without friction. At time $t = 0$, the string is in the shape of a parabola which is zero at both end-points with a maximum displacement of 1/4 metre (at the centre of the string). The initial velocity of each point of the string (except the left end) is 2 m/s downward. Write down the PDE, the boundary condition(s), and the initial condition(s) that $y(x,t)$ must satisfy. Include appropriate bounds on the variables. Do **NOT** solve the problem.

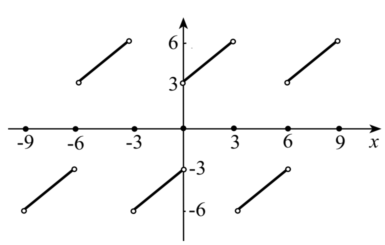
Answers: **1.** $-1/4$ **2.** -20π **3.** $-1/6$

4.(a) $a_{n+2} = \dfrac{-2(n-2)}{n+2}a_n,\ n \geq 0$ (b)$-1/\sqrt{2} < x < 1/\sqrt{2}$

5.(a) $a_1\left(x - \dfrac{x^3}{3}\right) + a_0\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^{n-1}(2n-3)(2n-1)}x^{2n}$ (b) $-\sqrt{3} < x < \sqrt{3}$

6.(a) $\dfrac{6}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{[1+2(-1)^{n+1}]}{n}\sin\frac{n\pi x}{3}$

(b)



7.(a) $8 + \dfrac{16}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-7(2n-1)^2\pi^2 t/16}\cos\frac{(2n-1)\pi x}{4}$

(b) Ends of the rod are insulated. (c) Limit is 8

8.

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2}, \quad 0 < x < 10, \quad t > 0, \qquad c^2 = \tau/\rho,\\
y(0,t) &= 0, \quad t > 0,\\
y_x(10,t) &= 0, \quad t > 0,\\
y(x,0) &= x(10-x)/100, \quad 0 < x < 10,\\
y_t(x,0) &= -2, \quad 0 < x < 10.
\end{aligned}$$