Sample Test 3 MATH3132

Time: 60 Minutes

1. Determine whether the functions $f(x) = x^3 - 4$ and $g(x) = x$ are orthogonal with respect to the weight function $w(x) = x$ on the intervals (a) $0 \le x \le 1$ and (b) $0 \le x \le 2$.

Answer: (a) Not orthogonal (b) Orthogonal

2. (a) Find the Fourier series of the function

$$f(x) = \begin{cases} 4, & 0 < x < 2 \\ 0, & 2 < x < 4 \end{cases} \qquad f(-x) = f(x), \qquad f(x+8) = f(x).$$

Simplify the series as much as possible.

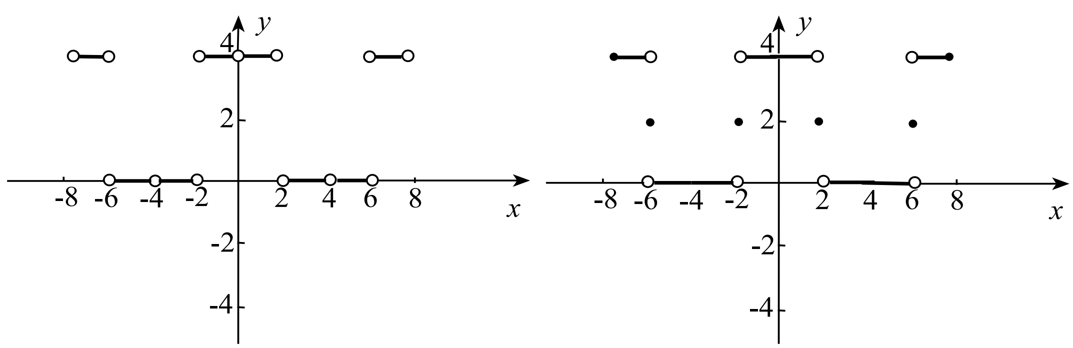
(b) On the interval $-8 \le x \le 8$, draw graphs of $f(x)$ and the function to which the Fourier series converges.

(c) Use your Fourier series to find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$.

Answer:

(a) $2 + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cos \frac{(2n-1)\pi x}{4}$

(b)



(c) $\pi/4$

3. Find eigenvalues and eigenfunctions of the Sturm-Liouville system

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \lambda y = 0, \quad 0 < x < L,$$
$$y(0) = 0,$$
$$y(L) = 0.$$

You may assume that all eigenvalues are greater than 1. Bonus: What is the weight function for the system?

Answer: $\lambda_n = 1 + n^2\pi^2/L^2$, where $n \ge 1$ is an integer $\quad y_n(x) = e^{-x} \sin \frac{n\pi x}{L} \quad w(x) = e^{2x}$

4. A rod with length 8 and thermal diffusivity 3 has insulated sides. It lies along the x-axis from $x = 0$ to $x = 8$. Initially, the temperature of its left end is 50° C and that of its right end is 100° C, and it rises linearly between the ends. The right end continues to be held at temperature 100° C, but its left end is insulated. Write down the partial differential equation satisfied by the temperature $U(x, t)$ at points in the rod and all boundary and initial conditions satisfied by $U(x, t)$. Include intervals on which each of these must be satisfied.

Answer:

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= 3\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < 8, \quad t > 0, \\
U_x(0, t) &= 0, \quad t > 0, \\
U(L, t) &= 100, \quad t > 0, \\
U(x, 0) &= 50 + 25x/4, \quad 0 < x < L.
\end{aligned}
$$