Midterm Examination #1 Solutions MATH3132

10 **1.** Evaluate the line integral

$$\int_C xy\,ds$$

where C is that part of the curve $x^2+y^2=4$, $x+z=4$ in the first octant from the point $(2,0,2)$ to the point $(0,2,4)$.

Parametric equations for the curve are

$$x=2\cos t,\quad y=2\sin t,\quad z=4-2\cos t,\quad 0\le t\le \pi/2.$$

The value of the line integral is

$$\int_C xy\,ds=\int_0^{\pi/2}(2\cos t)(2\sin t)\sqrt{(-2\sin t)^2+(2\cos t)^2+(2\sin t)^2}dt$$

$$=8\int_0^{\pi/2}\sin t\cos t\sqrt{1+\sin^2 t}dt=8\left\{\frac{1}{3}(1+\sin^2 t)^{3/2}\right\}_0^{\pi/2}=\frac{16\sqrt{2}-8}{3}.$$

Alternatively, with the parametric equations $x=-t$, $y=\sqrt{4-t^2}$, $z=4+t$, $-2\le t\le 0$,

$$\int_C xy\,ds=\int_{-2}^{0}-t\sqrt{4-t^2}\sqrt{(-1)^2+\left(\frac{-t}{\sqrt{4-t^2}}\right)^2+(1)^2}\,dt$$

$$=-\int_{-2}^{0}t\sqrt{4-t^2}\sqrt{2+\frac{t^2}{4-t^2}}\,dt=-\int_{-2}^{0}t\sqrt{8-t^2}\,dt$$

$$=-\left\{-\frac{1}{3}(8-t^2)^{3/2}\right\}_{-2}^{0}=\frac{16\sqrt{2}-8}{3}.$$

7 **2.** Evaluate the line integral

$$\int_C -\frac{y}{x^2}dx+\left(\frac{1}{x}+z\right)dy+(y-1)\,dz,$$

where C is the curve $x=z^2+1$, $y=z$ from $(1,0,0)$ to $(10,3,3)$.

Since $\nabla\left(\frac{y}{x}+yz-z\right)=-\frac{y}{x^2}\hat{\mathbf{i}}+\left(\frac{1}{x}+z\right)\hat{\mathbf{j}}+(y-1)\hat{\mathbf{k}}$ in the domain $x>0$, the line integral is independent of path in this domain. The value of the line integral is therefore

$$\int_C -\frac{y}{x^2}dx+\left(\frac{1}{x}+z\right)dy+(y-1)\,dz=\left\{\frac{y}{x}+yz-z\right\}_{(1,0,0)}^{(10,3,3)}=\frac{63}{10}.$$

3 **3.** Find a value for the constant c in order that the vector field

$$\mathbf{F}(x,y,z) = (cx^2 + y)\hat{\mathbf{i}} + x^3 z\hat{\mathbf{j}} + xyz^2\hat{\mathbf{k}}$$

have zero divergence at the point $(1,-2,3)$.

The divergence of the vector field is $\nabla\cdot\mathbf{F} = 2cx + 2xyz$. It is equal to zero at the point $(1,-2,3)$ if

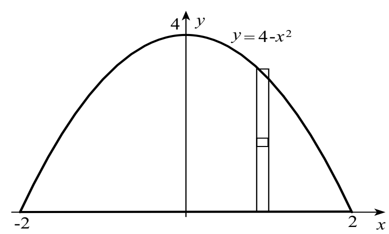
$$0 = 2c(1) + 2(1)(-2)(3) \qquad \Longrightarrow \qquad c = 6.$$

10 **4.** Evaluate the line integral

$$\oint_C (3x^2 - y^2 + 3ye^{3x})\,dx + (2x^2 + 1 + e^{3x})\,dy,$$

where C is the curve bounding the area enclosed by the curves $y = 4 - x^2$, $y = 0$.

If we use Green's theorem to evaluate the line integral, we get



$$I = \oint_C (3x^2 - y^2 + 3ye^{3x})\,dx + (2x^2 + 1 + e^{3x})\,dy$$
$$= -\iint_R (4x + 3e^{3x} + 2y - 3e^{3x})\,dA.$$

Due to the fact that x is an odd function, and R is symmetric about the y-axis, the integral of $4x$ is zero. Consequently,

$$I = -4\int_0^2 \int_0^{4-x^2} y\,dy\,dx = -4\int_0^2 \left\{\frac{y^2}{2}\right\}_0^{4-x^2} dx$$
$$= -2\int_0^2 (16 - 8x^2 + x^4)\,dx = -2\left\{16x - \frac{8x^3}{3} + \frac{x^5}{5}\right\}_0^2 = -\frac{512}{15}.$$

10 **5.** Evaluate the surface integral

$$\iint_S (x^2 + y^2)\,dS$$

where S is that part of the surface $2x + y - z = -1$ inside $x^2 + y^2 = 2$.

The projection of S onto the xy-plane is the interior of the circle $x^2 + y^2 = 2$, call it S_{xy}. Then,

$$\iint_S (x^2 + y^2)\,dS = \iint_{S_{xy}} (x^2 + y^2)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \iint_{S_{xy}} (x^2 + y^2)\sqrt{1 + 4 + 1}\,dA = \sqrt{6}\int_0^{2\pi}\int_0^{\sqrt{2}} r^2\,r\,dr\,d\theta$$
$$= \sqrt{6}\int_0^{2\pi} \left\{\frac{r^4}{4}\right\}_0^{\sqrt{2}} d\theta = \sqrt{6}\,\{\theta\}_0^{2\pi} = 2\sqrt{6}\pi.$$