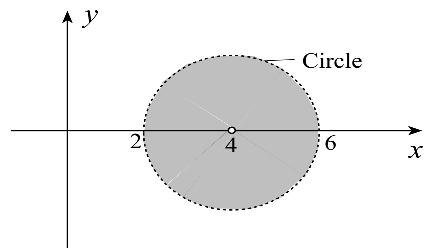
Midterm Examination #1 Math 3132 Engineering Mathematical Analysis 3 October 2, 2012

6 **1.** Let R be the shaded region of the xy-plane shown in the figure to the right.
(a) Find an algebraic expression describing the region.
(b) State whether the region is an open set, a closed set, a connected set, a domain, and a simply-connected domain.



(a) $0 < (x-4)^2 + y^2 < 4$
(b) The region is open, not closed, connected, a domain, but not a simply-connected domain.

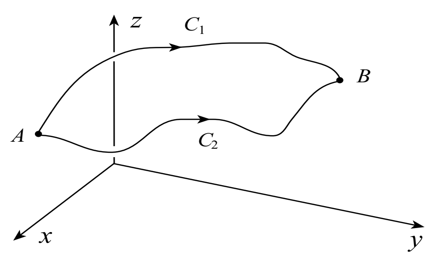
5 **2.** Let D be a simply-connected domain. Prove that if the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

around every closed curve in D, then the line integral is independent of path in D.

Let A and B be any two points in D, and let C_1 and C_2 be any two curves in D joining A to B. Since $C_1 - C_2$ is a closed curve, we can write that

$$0 = \int_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r}$$
$$= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$



This implies that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r},$$

and therefore the line intgral is independent of path in D.

8 **3.** Set up, but do **NOT** evaluate, a definite integral for the value of the line integral

$$\int_C (x^2 + yz) ds,$$

where C is the curve $x = y^2 + 1$, $z = x^2$ from the point $(10, 3, 100)$ to the point $(2, 1, 4)$. It is not necessary for you to simplify the integrand.

Since parametric equations for the curve are $x = t^2 + 1, \quad y = -t, \quad z = (t^2+1)^2, \quad -3 \le t \le -1$,

$$\int_C (x^2 + yz) ds = \int_{-3}^{-1} [(t^2+1)^2 - t(t^2+1)^2] \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$
$$= \int_{-3}^{-1} [(t^2+1)^2 - t(t^2+1)^2] \sqrt{(2t)^2 + (-1)^2 + [4t(t^2+1)]^2}\, dt.$$

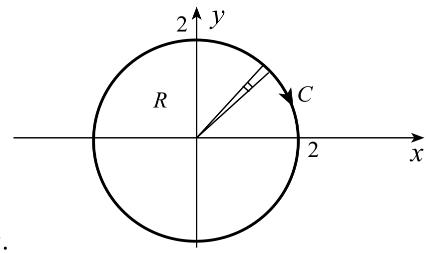
12 **4.** Evaluate the line integral

$$\oint_C -x^2 y\,dx + xy^2\,dy$$

once around the curve $x^2 + y^2 = 4$.

By Green's theorem,

$$\begin{aligned}\oint_C -x^2 y\,dx + xy^2\,dy &= -\iint_R (y^2 + x^2)\,dA \\ &= -\int_0^{2\pi}\int_0^2 (r^2) r\,dr\,d\theta \\ &= -\int_0^{2\pi} \left\{\frac{r^4}{4}\right\}_0^2 d\theta = -4\,\{\theta\}_0^{2\pi} = -8\pi.\end{aligned}$$



9 **5.** Set up, but do **NOT** evaluate, a double iterated integral for the value of the surface integral

$$\iint_S (2xy - z)dS$$

where S is that part of the surface $z = 10 - x^2 - y^2$ above the plane $z = 1$.

$$\begin{aligned}\iint_S (2xy - z)dS &= \iint_{S_{xy}} (2xy - z)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA \\ &= \iint_{S_{xy}} (2xy - z)\sqrt{1 + (-2x)^2 + (-2y)^2}\,dA \\ &= \iint_{S_{xy}} (2xy - 10 + x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\,dA \\ &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (2xy - 10 + x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\,dy\,dx,\end{aligned}$$

or,

$$\int_0^{2\pi}\int_0^3 (2r^2\cos\theta\sin\theta - 10 + r^2)\sqrt{1 + 4r^2}\,r\,dr\,d\theta.$$