8 **1.** Evaluate the line integral

$$\int_C (y-x)\,ds,$$

where C is the curve $y = 2x$, $z + x^2 = 2$ from the point $(2, 4, -2)$ to the point $(-1, -2, 1)$.

With parametric equations $x = -t$, $y = -2t$, $z = 2 - t^2$, $-2 \le t \le 1$,

$$\int_C (y-x)\,ds = \int_{-2}^{1} (-2t+t)\sqrt{1+4+4t^2}\,dt = -\int_{-2}^{1} t\sqrt{5+4t^2}\,dt = -\left\{\frac{1}{12}(5+4t^2)^{3/2}\right\}_{-2}^{1}$$

$$= -\frac{1}{12}(27 - 21\sqrt{21}) = \frac{1}{4}(7\sqrt{21} - 9).$$

8 **2.** Evaluate the line integral

$$\oint_C (x^3y^2 - 3x)\,dx + (y\sin x - x)\,dy,$$

where C is the curve bounding the area enclosed by the curves $y = 1 - x^2$, $y = x^2 - 1$.

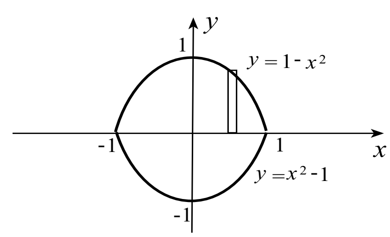
Using Green's Theorem,

$$\oint_C (x^3y^2 - 3x)\,dx + (y\sin x - x)\,dy$$

$$= -\iint_R (y\cos x - 1 - 2x^3y)\,dA$$

$$= 4\int_0^1 \int_0^{1-x^2} dy\,dx = 4\int_0^1 (1-x^2)\,dx$$

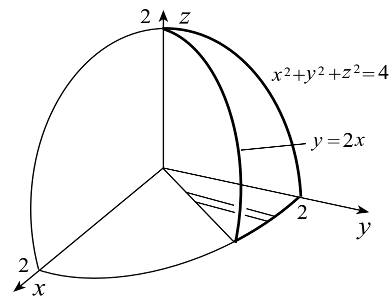
$$= 4\left\{x - \frac{x^3}{3}\right\}_0^1 = \frac{8}{3}$$



10 **3.** Set up, but do **NOT** evaluate, a double iterated integral for the value of the surface integral

$$\iint_S (x^2z^2\hat{\mathbf{i}} + yz\hat{\mathbf{j}} - x\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS,$$

where S is the smaller part of $x^2 + y^2 + z^2 = 4$ in the first octant cut out be the plane $y = 2x$, and $\hat{\mathbf{n}}$ is the unit upper normal to S.



Let S_{xy} be the projection of the surface onto the xy-plane. A normal to the surface is $(2x, 2y, 2z)$, and therefore $\hat{\mathbf{n}} = \dfrac{(x,y,z)}{2}$. Since $2x + 2z\dfrac{\partial z}{\partial x} = 0$, we obtain $\dfrac{\partial z}{\partial x} = -\dfrac{x}{z}$. Similarly, $\dfrac{\partial z}{\partial y} = -\dfrac{y}{z}$.
Thus,

$$\begin{aligned}
\iint_S (x^2z^2\hat{\mathbf{i}} + yz\hat{\mathbf{j}} - x\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS &= \iint_{S_{xy}} \frac{1}{2}(x^3z^2 + y^2z - xz)\sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\, dA \\
&= \frac{1}{2}\iint_{S_{xy}} (x^3z^2 + y^2z - xz)\frac{2}{z}\, dA = \iint_{S_{xy}} (x^3z + y^2 - x)\, dA \\
&= \int_0^{2/\sqrt{5}} \int_{2x}^{\sqrt{4-x^2}} (x^3\sqrt{4 - x^2 - y^2} + y^2 - x)\, dy\, dx.
\end{aligned}$$

14 **4.** Evaluate the line integral

$$\oint_C (x^2 - y)\, dx - (y + xz^2)\, dy + x^2\, dz,$$

where C is the curve $x^2 + y^2 = 1$, $y + z = 1$, directed counterclockwise as viewed from the origin.

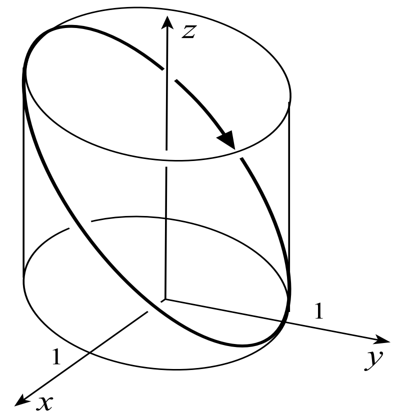
Let us choose S as that part of the plane $y + z = 1$ inside C. If $\mathbf{F} = (x^2 - y)\hat{\mathbf{i}} - (y + xz^2)\hat{\mathbf{j}} + x^2\hat{\mathbf{k}}$, then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y & -y - xz^2 & x^2 \end{vmatrix}$$

$$= 2xz\,\hat{\mathbf{i}} - 2x\,\hat{\mathbf{j}} + (1 - z^2)\hat{\mathbf{k}}.$$

With $\hat{\mathbf{n}} = \dfrac{(0, -1, -1)}{\sqrt{2}}$,

Stokes's Theorem gives



$$\oint_C (x^2 - y)\, dx - (y + xz^2)\, dy + x^2\, dz = \iint_S \frac{2x - 1 + z^2}{\sqrt{2}}\, dS = \frac{1}{\sqrt{2}} \iint_S [2x - 1 + (1 - y)^2]\, dS$$

$$= \frac{1}{\sqrt{2}} \iint_{S_{xy}} (2x - 2y + y^2)\sqrt{1 + 1} dA = \iint_{S_{xy}} y^2\, dA$$

$$= \int_0^{2\pi} \int_0^1 r^2 \sin^2\theta\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left\{ \frac{r^4}{4} \sin^2\theta \right\}_0^1 d\theta = \frac{1}{4} \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \frac{1}{8} \left\{ \theta - \frac{1}{2} \sin 2\theta \right\}_0^{2\pi} = \frac{\pi}{4}.$$