3132 Midterm Examination 1 Fall 2024 Solutions

1. Evaluate the line integral $\displaystyle\int_C 2xyz\,dx + \left(\frac{-1}{y^2} + x^2z + 2y\right)dy + (x^2y + 3z^2)\,dz$ where C is the longer part of the curve $x^2 + y^2 - 4y = -3$, $y + z = 4$ from $(0,1,3)$ to $(1,2,2)$.

Since

$$\nabla\left(x^2yz + \frac{1}{y} + y^2 + z^3\right) = 2xyz\hat{\mathbf{i}} + \left(x^2z - \frac{1}{y^2} + 2y\right)\hat{\mathbf{j}} + (x^2y + 3z^2)\hat{\mathbf{k}},$$

the line integral is independent of path in any domian that does not contain points in the xz-plane where $y = 0$. If we set $y = 0$ in the equation $x^2 + y^2 - 4y = -3$, we obtain $x^2 = -3$, an impossibility. The curve does not therefore pass through the xz-plane. The value of the line integral is

$$\int_C 2xyz\,dx + \left(\frac{-1}{y^2} + x^2z + 2y\right)dy + (x^2y + 3z^2)\,dz = \left\{x^2yz + \frac{1}{y} + y^2 + z^3\right\}_{(0,1,3)}^{(1,2,2)}$$

$$= \left(4 + \frac{1}{2} + 4 + 8\right) - (1 + 1 + 27) = -\frac{25}{2}.$$

5 **2.** With Stokes's theorem line integrals can sometimes be replaced with surface integrals. With the divergence theorem, surface integrals can somtimes be replaced with triple integrals. Does this mean that sometimes line integrals can be replaced by triple integrals? Explain.

No. The divergence theorem can replace a surface integral (of a special form) over a closed surface with a triple integral. Stokes's theorem replaces a line integral over a closed curve with a surface integral over a surface with the curve as its edge, but the surface is never closed. Hence, we cannot get from a line integral to a triple integral.

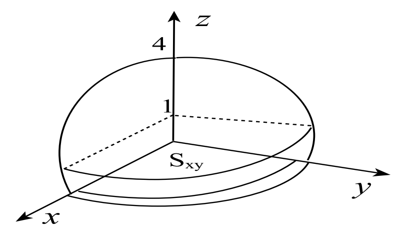
15 **3.** Set up, but do NOT evaluate, a double iterated integral in polar coordinates for the surface integral

$$\iint_S (x + y^2 z)\, dS$$

where S is that part of $z = 4 - x^2 - y^2$ above $z = 1$. Simplify the integrand as much as possible without doing any integration.

If we project the surface onto the area S_{xy} in the xy-plane,

$$\begin{aligned} dS &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \sqrt{1 + (-2x)^2 + (-2y)^2}\, dA \\ &= \sqrt{1 + 4x^2 + 4y^2}\, dA. \end{aligned}$$



Then,

$$\iint_S (x + y^2 z)\, dS = \iint_{S_{xy}} (x + y^2(4 - x^2 - y^2)\sqrt{1 + 4x^2 + 4y^2}\, dA.$$

Since S_{xy} is symmetric about the y-axis, and x is an odd function, integration of this term will be zero. Hence,

$$\begin{aligned} \iint_S (x + y^2 z)\, dS &= \int_{-\pi}^{\pi} \int_0^{\sqrt{3}} r^2 \sin^2\theta (4 - r^2)\sqrt{1 + 4r^2}\, r\, dr\, d\theta \\ &= \int_{-\pi}^{\pi} \int_0^{\sqrt{3}} r^3 \sin^2\theta (4 - r^2)\sqrt{1 + 4r^2}\, dr\, d\theta. \end{aligned}$$

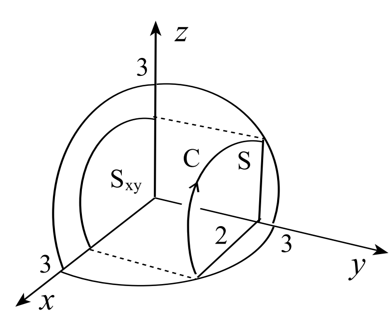
20 **4.** Evaluate

$$\oint_C (y\hat{\mathbf{i}} + x^2 y\hat{\mathbf{j}} + x^3\hat{\mathbf{k}}) \cdot d\mathbf{r}$$

where C is the curve $x^2 + y^2 + z^2 = 9$, $y = 2$ directed counterclockwise as viewed from the origin.

Let S be that part of the plane $= 2$ inside C. By Stokes's theorem,

$$\begin{aligned} &\oint_C (y\hat{\mathbf{i}} + x^2 y\hat{\mathbf{j}} + x^3\hat{\mathbf{k}}) \cdot d\mathbf{r} \\ &\qquad = \iint_S \nabla \times (y, x^2 y, x^3) \cdot \hat{\mathbf{n}}\, dS, \end{aligned}$$



where

$$\nabla \times (x, x^2y, x^3) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & x^2y & x^3 \end{vmatrix} = -3x^2\hat{\mathbf{i}} + (2xy-1)\hat{\mathbf{k}}.$$

Since $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$, and $dS = dA$,

$$\begin{aligned}\oint_C (y\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}} + x^3\hat{\mathbf{k}}) \cdot d\mathbf{r} &= \iint_S 3x^2\, dS = 3\iint_{S_{xz}} x^2\, dA = 3\int_{-\pi}^{\pi}\int_0^{\sqrt{5}} r^2 \sin^2\theta\, r\, dr\, d\theta \\ &= 3\int_{-\pi}^{\pi} \left\{\frac{r^4}{4}\sin^2\theta\right\}_0^{\sqrt{5}} d\theta = \frac{75}{4}\int_{-\pi}^{\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta \\ &= \frac{75}{8}\left\{\theta - \frac{1}{2}\sin 2\theta\right\}_{-\pi}^{\pi} = \frac{75\pi}{4}.\end{aligned}$$