Midterm Examination #1 Math 3132 July 11, 2017 60 minutes

7 **1.** (a) Find all domains in which the line

$$\int_C \frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2}$$

is independent of path. Justify your answer.

(b) Evaluate the line integral along that part of $x^2 + y^2 + z^2 = 4, \quad y = x$ in the first octant from $(\sqrt{2}, \sqrt{2}, 0)$ to $(0, 0, 2)$.

(a) Since $\nabla \left[\frac{1}{2} \ln{(x^2 + y^2 + z^2)}\right] = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{x^2 + y^2 + z^2}$, the line integral is independent of path in any domain that does not contain the origin $(0, 0, 0)$.

(b) The value of the line integral is

$$\left\{\frac{1}{2} \ln{(x^2 + y^2 + z^2)}\right\}_{(\sqrt{2},\sqrt{2},0)}^{(0,0,2)} = \frac{1}{2}(\ln 4 - \ln 4) = 0.$$

14 **2.** Evaluate the line integral

$$\oint_C (x\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} - y^2\hat{\mathbf{k}}) \cdot d\mathbf{r},$$

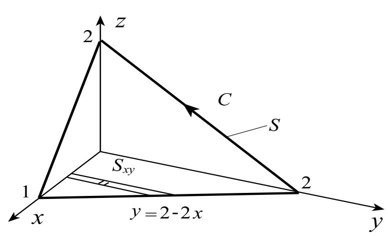
where C is the triangle bounding that part of $2x + y + z = 2$ in the first octant, directed clockwise as viewed from the origin.

Let S be that part of the plane $2x + y + z = 2$ in the first octant.

$$\nabla \times (x, y^2, -y^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y^2 & -y^2 \end{vmatrix} = (-2y, 0, 0).$$

Stokes's theorem gives

$$\begin{aligned}
\oint_C (x\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} - y^2\hat{\mathbf{k}}) \cdot d\mathbf{r} &= \iint_S (-2y, 0, 0) \cdot \frac{(2,1,1)}{\sqrt{6}}\, dS \\
&= \iint_S -\frac{4y}{\sqrt{6}}\, dS = -\frac{4}{\sqrt{6}} \iint_{S_{xy}} y\sqrt{1 + (-2)^2 + (-1)^2}\, dA \\
&= -4\int_0^1 \int_0^{2-2x} y\, dy\, dx = -4\int_0^1 \left\{\frac{y^2}{2}\right\}_0^{2-2x} dx \\
&= -2\int_0^1 (2 - 2x)^2)\, dx = -2\left\{-\frac{1}{6}(2 - 2x)^3\right\}_0^1 = -\frac{8}{3}.
\end{aligned}$$

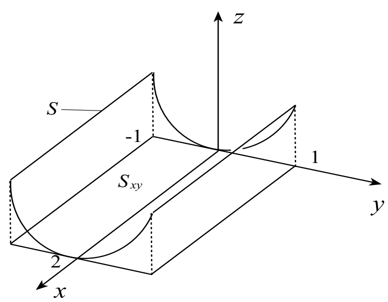


7 **3.** Set up, but do **NOT** evaluate, a double iterated integral for the value of the surface integral

$$\iint_S (x^2 + z)\, dS,$$

where S is that part of $z = y^2$ bounded by the planes $x = 1$, $x = 0$, and $z = 1$.

$$\iint_S (x^2 + z)\, dS = \iint_{S_{xy}} (x^2 + y^2)\sqrt{1 + (2y)^2}\, dA = \int_0^1 \int_{-1}^1 (x^2 + y^2)\sqrt{1 + 4y^2}\, dy\, dx$$



12 **4.** Evaluate the surface integral

$$\oiint_S (x^2\hat{\mathbf{i}} - y^2\hat{\mathbf{j}} - (x^2+y^2)z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS,$$

where S is the surface $x^2+y^2+z^2=a^2$, ($a>0$ a constant), and $\hat{\mathbf{n}}$ is the unit inward pointing normal to the surface.

By the divergence theorem,

$$\begin{aligned}
\oiint_S (x^2\hat{\mathbf{i}} - y^2\hat{\mathbf{j}} - (x^2+y^2)z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS &= -\iiint_V [2x - 2y - (x^2+y^2)]\, dV \\
&= \iiint_V (x^2+y^2)\, dV \\
&= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a R^2\sin^2\phi\, R^2\sin\phi\, dR\, d\phi\, d\theta \\
&= 8\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{R^5}{5}\right\}_0^a \sin^3\phi\, d\phi\, d\theta \\
&= \frac{8a^5}{5}\int_0^{\pi/2}\int_0^{\pi/2} \sin\phi(1-\cos^2\phi)\, d\phi\, d\theta \\
&= \frac{8a^5}{5}\int_0^{\pi/2}\left\{-\cos\phi + \frac{1}{3}\cos^3\phi\right\}_0^{\pi/2} d\theta \\
&= \frac{8a^5}{5}\left(\frac{2}{3}\right)\{\theta\}_0^{\pi/2} = \frac{8\pi a^5}{15}.
\end{aligned}$$