Midterm Examination #2 Solutions MATH3132

1. Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

where $\mathbf{F} = x^2 z\hat{\mathbf{i}} - (x+3)\hat{\mathbf{j}} + \hat{\mathbf{k}}$, S is that part of the surface $z = 4 - x^2 - y^2$ above the xy-plane, and $\hat{\mathbf{n}}$ is the unit upward normal to the surface.

The unit upward normal to the surface is $\hat{\mathbf{n}} = \dfrac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}}$. Consequently,

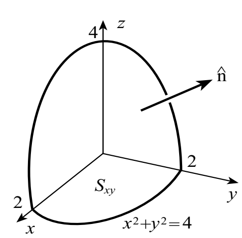
$$\mathbf{F} \cdot \hat{\mathbf{n}} = \frac{2x^3 z - 2xy - 6y + 1}{\sqrt{4x^2 + 4y^2 + 1}}.$$

If we project the surface onto the circle $S_{xy} : x^2 + y^2 \le 4$ in the xy-plane (figure below), then

$$\begin{aligned}\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \iint_{S_{xy}} \frac{2x^3 z - 2xy - 6y + 1}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{1 + (2x)^2 + (2y)^2}\, dA \\ &= \iint_{S_{xy}} [2x^3(4 - x^2 - y^2) - 2xy - 6y + 1]\, dA.\end{aligned}$$

Because $8x^3 - 2x^5 - 2x^3y^2 - 2xy$ is an odd function of x, and S_{xy} is symmetric about the y-axis, the integral of these terms is equal to zero. Since $-6y$ is an odd function of y, and S_{xy} is symmetric about the x-axis, the integral of this term is also zero. This leaves

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{S_{xy}} 1\, dA = \text{Area of } S_{xy} = 4\pi.$$



2. You are to evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

once around the curve C: $x^2 + y^2 + z^2 = 3$, $2z = x^2 + y^2$ directed countercclockwise as viewed from the origin. You are told that the curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = (x^2 + y^2)\hat{\mathbf{i}} - xz^2\hat{\mathbf{j}} + z^2 e^{x^2}\hat{\mathbf{k}}.$$

Set up, but do **NOT** evaluate, a double iterated integral that has the same value as the line integral.

The curve is the circle $x^2 + y^2 = 2$ in the plane $z = 1$ (figure below). If we use Stokes's theorem to evaluate the line integral and choose S as that part of the plane $z = 1$ inside C, then

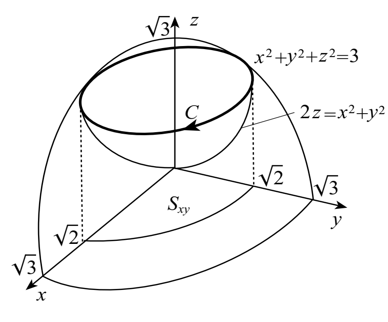
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS.$$

Since $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S -z^2 e^{x^2}\, dS = -\iint_{S_{xy}} (1)e^{x^2}\, dA,$$

where S_{xy}, the projection of S onto the xy-plane is the circle $x^2 + y^2 \le 2$. If we use polar coordinates,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = -\int_0^{2\pi} \int_0^{\sqrt{2}} e^{r^2 \cos^2\theta}\, r\, dr\, d\theta.$$



3. (a) Find all real singular points for the differential equation

$$(x^2+1)y'' - 2y = 0.$$

(b) Determine, with justification, the radius of convergence of the Maclaurin series solution of the differential equation.

(c) Find the Maclaurin series solution of the differential equation. Write your answer in sigma notation, simplified as much as possible.

(a) Since the function $\dfrac{-2}{x^2+1}$ has a convergent Taylor series about any value of x, the differential equation has no real singularities.
(b) Since the differential equation has complex singularities $x = \pm i$, a minimum value for the radius of convergence is 1 (the distance from $z = 0$ to $z = \pm i$). Because coefficients are polynomials, this is the actual radius of convergence of the Maclaurin series solution.
(c) If we assume a Maclaurin series solution $\sum_{n=0}^{\infty} a_n x^n$ with known radius of convergence $R = 1$, we get

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} -2a_n x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} \\
&= \sum_{n=0}^{\infty} -2a_n x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2} x^n \\
&= \sum_{n=0}^{\infty} -2a_n x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n \\
&= \sum_{n=0}^{\infty} [-2a_n + n(n-1)a_n + (n+2)(n+1)a_{n+2}]x^n.
\end{aligned}$$

When we equate coefficients of powers of x to zero, we obtain

$$-2a_n + n(n-1)a_n + (n+2)(n+1)a_{n+2} = 0,$$

and this implies that

$$a_{n+2} = \frac{[-n(n-1)+2]a_n}{(n+2)(n+1)} = -\frac{(n-2)}{n+2}a_n, \quad n \geq 0.$$

Iteration gives $a_2 = a_0$, and $0 = a_4 = a_6 = \ldots$, and

$$\begin{aligned}
n = 1: &\quad a_3 = \frac{1}{3}a_1, \\
n = 3: &\quad a_5 = -\frac{1}{5}a_3 = -\frac{1}{3\cdot 5}a_1, \\
n = 5: &\quad a_7 = -\frac{3}{7}a_5 = \frac{1}{5\cdot 7}a_1, \\
n = 7: &\quad a_9 = -\frac{5}{9}a_7 = -\frac{1}{7\cdot 9}a_1.
\end{aligned}$$

The solution is therefore

$$\begin{aligned}
y(x) &= a_0(1+x^2) + a_1\left(x + \frac{x^3}{3} - \frac{x^5}{3\cdot 5} + \frac{x^7}{5\cdot 7} - \frac{x^9}{7\cdot 9} + \cdots\right) \\
&= a_0(1+x^2) + a_1 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)} x^{2n+1}.
\end{aligned}$$