Midterm Examination #2 MATH3132 Mathematical Methods for Engineers 3

Time: 60 Minutes

Student Name (Print): ______________________________

Student Signature: ______________________________

Student Number: ______________________________

INSTRUCTIONS:

1. No aids permitted.

2. Check that your examination has 3 questions.

6 **1.** Find all singular points for the differential equation

$$x^2(x-1)\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + xy = 0,$$

and determine whether they are regular or irregular singular points. Justify your answers.

Consider the functions

$$\frac{3}{x^2(x-1)} \qquad \text{and} \qquad \frac{x}{x^2(x-1)} = \frac{1}{x(x-1)}.$$

Since neither of these functions has a Maclaurin series nor a Taylor series about $x = 1$, the points $x = 0$ and $x = 1$ are singular. To determine whether $x = 0$ is regular or irregular singular, consider the functions

$$\frac{3x}{x^2(x-1)} = \frac{3}{x(x-1)} \qquad \text{and} \qquad \frac{x^2}{x(x-1)} = \frac{x}{x-1}.$$

Since the first of these does not have a Maclaurin series, $x = 0$ is irregular singular. To discuss $x = 1$, consider

$$\frac{3(x-1)}{x^2(x-1)} = \frac{3}{x^2} \qquad \text{and} \qquad \frac{(x-1)^2}{x(x-1)} = \frac{x-1}{x}.$$

Since both of these functions have Taylor series about $x = 1$, $x = 1$ is regular singular.

14 **2.** Evaluate the surface integral

$$\oiint_S [x^2z\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} + (xy + 3x^2z)\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS$$

where S is the surface surrounding the volume bounded by the surfaces $x^2 + y^2 = 4$, $z = 0$ and $z = 3$, and $\hat{\mathbf{n}}$ is the unit inward normal to the surface.

If V is the volume enclosed by the surface, then the divergence theorem says that

$$\oiint_S [x^2z\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} + (xy + 3x^2z)\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS = -\iiint_V (2xz + 3y^2 + 3x^2)\, dV.$$

Since the function $2xz$ is an odd function of x, and V is symmetric about the yz-plane, it integrates to zero. When we use cylindrical coordinates on the remaining terms,

$$\begin{aligned}
\oiint_S [x^2z\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} + (xy + 3x^2z)\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS &= -3\int_{-\pi}^{\pi}\int_0^2\int_0^3 r^3\, dz\, dr\, d\theta \\
&= -3\int_{-\pi}^{\pi}\int_0^2 \left\{r^3 z\right\}_0^3 dr\, d\theta = -9\int_{-\pi}^{\pi}\int_0^2 r^3\, dr\, d\theta \\
&= -9\int_{-\pi}^{\pi} \left\{\frac{r^4}{4}\right\}_0^2 d\theta = -36\int_{-\pi}^{\pi} d\theta = -36\left\{\theta\right\}_{-\pi}^{\pi} = -72\pi.
\end{aligned}$$

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20 **3.** (a) Find the Maclaurin series solution of the differential equation

$$(1+4x^2)\frac{d^2y}{dx^2} - 8y = 0.$$

(b) Is your solution a general one? Give reasons why you should expect this.
(c) Use the formula

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

to find the radius of convergence of any series in your answer to part (a). Is this what should be expected? Explain.

(a) If we assume a Maclaurin series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with positive radius of convergence, then

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} 4n(n-1)a_n x^n + \sum_{n=0}^{\infty} -8a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 4n(n-1)a_n x^n + \sum_{n=0}^{\infty} -8a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + 4n(n-1)a_n - 8a_n]x^n. \end{aligned}$$

When we equate coefficients to zero, we obtain

$$(n+2)(n+1)a_{n+2} + 4n(n-1)a_n - 8a_n = 0,$$

from which

$$a_{n+2} = -\frac{4n(n-1)-8}{(n+2)(n+1)}a_n = -\frac{4(n^2-n-2)}{(n+2)(n+1)}a_n = -\frac{4(n-2)(n+1)}{(n+2)(n+1)} = -\frac{4(n-2)}{n+2}a_n, \quad n \geq 0.$$

For $n=0$: $a_2 = \frac{4\cdot 2}{2}a_0.$
For $n=2$: $a_4 = 0$ and therefore $a_6 = a_8 = \cdots = 0.$
For $n=1$: $a_3 = \frac{4(1)}{3}a_1.$
For $n=3$: $a_5 = -\frac{4}{5}a_3 = -\frac{4^2}{3\cdot 5}a_1.$
For $n=5$: $a_7 = -\frac{4(3)}{7}a_5 = \frac{4^3\cdot 3}{3\cdot 5\cdot 7}a_1.$
For $n=7$: $a_9 = -\frac{4(5)}{9}a_7 = -\frac{4^4\cdot 3\cdot 5}{3\cdot 5\cdot 7\cdot 9}a_1.$
The solution is

$$y(x) = a_0 + 4a_0x^2 + a_1\left[x + \frac{4}{3}x^3 - \frac{4^2}{3\cdot 5}x^5 + \frac{4^3\cdot 3}{3\cdot 5\cdot 7}x^7 - \frac{4^4\cdot 3\cdot 5}{3\cdot 5\cdot 7\cdot 9}x^9 + \cdots\right]$$

$$= a_0(1+4x^2) + a_1\left[x + \frac{4}{3}x^3 - \frac{4^2}{3\cdot 5}x^5 + \frac{4^3}{5\cdot 7}x^7 - \frac{4^4}{7\cdot 9}x^9 + \cdots\right]$$

$$= a_0(1+4x^2) + a_1\sum_{n=0}^{\infty}\frac{(-1)^{n+1}4^n}{(2n-1)(2n+1)}x^{2n+1}.$$

(b) Since the solution has two arbitrary constants, it is a general solution. We should expect this because $x = 0$ is an ordinary point.
(c) If we set $z = x^2$, the series in part (a) becomes

$$x\sum_{n=0}^{\infty}\frac{(-1)^{n+1}4^n}{(2n-1)(2n+1)}z^n.$$

The radius of convergence of the series is

$$R_z = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}4^n}{(2n-1)(2n+1)}}{\frac{(-1)^{n+2}4^{n+1}}{(2n+1)(2n+3)}}\right| = \frac{1}{4}.$$

Hence $R_x = 1/2$. Singularities of the differential equation are $x = \pm i/2$. The distance from $x = 0$ to each of these is $1/2$, and this should be the radius of convergence.