3132 Solutions to Midterm Examination #2 Fall 2014

8 **1.** Find all singular points for the differential equation

$$x^2(x+2)^2\frac{d^2y}{dx^2} + 3x\sin x\frac{dy}{dx} + 2y = 0,$$

and determine whether they are regular or irregular singular points. Justify your answers.

Consider the functions

$$\frac{3x\sin x}{x^2(x+2)^2} = \frac{3\sin x}{x(x+2)^2}, \qquad \frac{2}{x^2(x+2)^2}.$$

Since the first of these does not have a Taylor series about $x = -2$ and the second does not have a Maclaurin series or a Taylor series about $x = -2$, the points $x = 0$ and $x = -2$ are singular.

To discuss $x = 0$, consider the functions

$$\frac{3x\sin x}{x(x+2)^2} = \frac{3\sin x}{(x+2)^2}, \qquad \frac{2x^2}{x^2(x+2)^2} = \frac{2}{(x+2)^2}.$$

Since both have convergent Maclaurin series, $x = 0$ is regular singular.

To discuss $x = -2$, consider the function

$$\frac{3(x+2)\sin x}{x(x+2)^2} = \frac{3\sin x}{x(x+2)}.$$

Since it does not have a Taylor series about $x = -2$, $x = -2$ is irregular singular.

16 **2.** Find the Frobenius solution about $x = 0$ for the differential equation

$$x^2\frac{d^2y}{dx^2} - x(1+x)\frac{dy}{dx} + y = 0.$$

Express your solution in sigma notation simplified as much as possible. If possible, find sums for any series in your solution.

If we assume $y = \displaystyle\sum_{n=0}^{\infty} a_n x^{n+r}$, and substitute into the differential equation,

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} -(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} -(n+r)a_n x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} -(n+r)a_n x^{n+r} + \sum_{n=1}^{\infty} -(n+r-1)a_{n-1} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r}.
\end{aligned}$$

The indicial equation is

$$0 = r(r-1) - r + 1 = r^2 - 2r + 1 = (r-1)^2, \qquad \text{with root } r = 1.$$

The remaining coefficients give

$$(n+1)na_n - (n+1)a_n - na_{n-1} + a_n = 0, \quad n \geq 1,$$

from which

$$a_n = \frac{na_{n-1}}{(n+1)n - (n+1) + 1} = \frac{a_{n-1}}{n}.$$

Iterating:
For $n = 1$, $a_1 = a_0$.
For $n = 2$, $a_2 = \dfrac{a_1}{2} = \dfrac{a_0}{2}$.
For $n = 3$, $a_3 = \dfrac{a_2}{3} = \dfrac{a_0}{3 \cdot 2}$.
For $n = 4$, $a_4 = \dfrac{a_3}{4} = \dfrac{a_0}{4!}$.
The solution is

$$y(x) = x\left(a_0 + a_0 x + \frac{a_0}{2}x^2 + \frac{a_0}{3!}x^3 + \cdots\right) = x\sum_{n=0}^{\infty}\frac{x^n}{n!} = a_0 x e^x.$$

10 **3.** Find the Fourier series for the function

$$f(x) = \begin{cases} 1-2x, & -1 < x \le 0 \\ 1+2x, & 0 < x \le 1, \end{cases} \qquad f(x+2) = f(x).$$

Simplify coefficients as much as possible.

Since the function is even, the Fourier series will be a Fourier cosine series with

$$a_0 = \frac{1}{1}\int_0^1 (1+2x)\,dx = 2\left\{x + x^2\right\}_0^1 = 4.$$

For $n \ge 1$,

$$a_n = \frac{2}{1}\int_0^1 (1+2x)\cos n\pi x\,dx = 2\left[\left\{\frac{1}{n\pi}(1+2x)\sin n\pi x\right\}_0^1 - \int_0^1 \frac{2}{n\pi}\sin n\pi x\,dx\right]$$

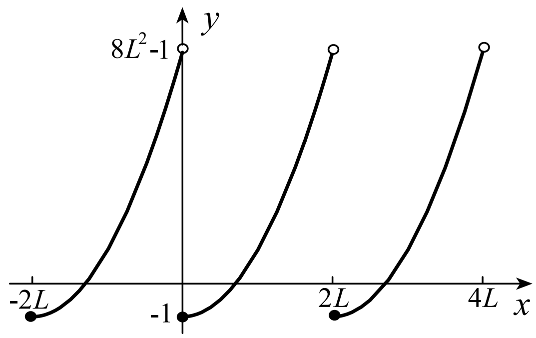
$$= -\frac{4}{n\pi}\left\{-\frac{1}{n\pi}\cos n\pi x\right\}_0^1 = \frac{4}{n^2\pi^2}[\cos n\pi - 1] = \frac{4}{n^2\pi^2}[(-1)^n - 1].$$

The Fourier cosine series is therefore

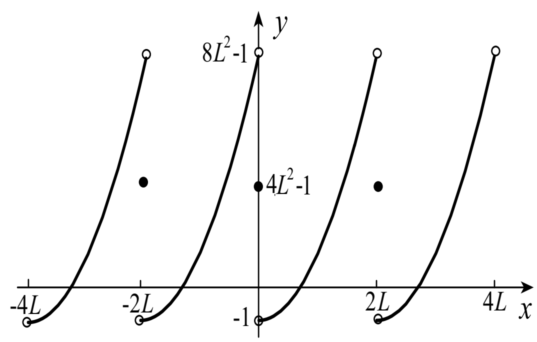
$$2 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}[(-1)^n - 1]\cos n\pi x = 2 - \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)\pi x.$$

6 **4.** The Fourier series for the periodic function $f(x)$ shown below is

$$\frac{8L^2}{3} - 1 + \frac{8L^2}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \cos \frac{n\pi x}{L} - \frac{\pi}{n} \sin \frac{n\pi x}{L} \right).$$



(a) On the axes below, draw a graph of the function to which the Fourier series converges.



(b) Use the Fourier series to find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

If we set $x = 0$ in the series and equate the result to $4L^2 - 1$. we obtain

$$4L^2 - 1 = \frac{8L^2}{3} - 1 + \frac{8L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8L^2} \left(4L^2 - 1 - \frac{8L^2}{3} + 1 \right) = \frac{\pi^2}{6}.$$