Midterm Examination 2 for Math 3132 July 25, 2017

Student Number **Student Name**

8 **1.** Suppose that a solution of the differential equation

$$(x^2-1)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 5y = 0.$$

is assumed to have a Maclaurin series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

with positive radius of convergence. Find a recurrence relation for the coefficients a_n, simplified as much as possible. Do **NOT** iterate this formula; that is, do **NOT** solve for the a_n.

When we substitute the series into the differential equation,

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} -n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} -3na_n x^n + \sum_{n=0}^{\infty} -5a_n x^n \\
&= \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} -(n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} -3na_n x^n + \sum_{n=0}^{\infty} -5a_n x^n \\
&= \sum_{n=0}^{\infty} [n(n-1)a_n - (n+2)(n+1)a_{n+2} - 3na_n - 5a_n]x^n.
\end{aligned}$$

When we equate coefficients to zero,

$$n(n-1)a_n - (n+2)(n+1)a_{n+2} - 3na_n - 5a_n = 0, \quad n \geq 0.$$

We now solve for a_{n+2},

$$a_{n+2} = -\frac{3na_n + 5a_n - n(n-1)a_n}{(n+2)(n+1)} = \frac{n^2-4n-5}{(n+2)(n+1)}a_n = \frac{(n-5)(n+1)}{(n+2)(n+1)}a_n = \frac{n-5}{n+2}a_n.$$

8 **2.** When a solution of the differential equation

$$(x^2+3)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y=0$$

is represented in terms of its Maclaurin series, $y(x)=\sum_{n=0}^{\infty} a_n x^n$, its coefficients a_n must satisfy the recurrence relation

$$a_{n+2}=\frac{-(n-1)}{3(n+1)}a_n, \quad n \geq 0.$$

You may assume this relationship, Do **NOT** prove it.

(a) Use the recurrence relation to find a general solution of the differential equation. Express your answer in sigma notation, simplified as much as possible.

(b) Determine the radius of convergence of the series solution?

(a) For $n=0$, $a_2=\frac{a_0}{3}$.
For $n=1$, $a_3=0$. This implies that $0=a_5=a_7\cdots$.
For $n=2$, $a_4=-\frac{1}{3\cdot 3}a_2=-\frac{a_0}{3\cdot 3\cdot 3}$.
For $n=4$, $a_6=\frac{-3}{3\cdot 5}a_4=\frac{3a_0}{3^4\cdot 5}$.
For $n=6$, $a_8=\frac{-5}{3\cdot 7}a_6=-\frac{3\cdot 5a_0}{3^5\cdot 5\cdot 7}$.
For $n=8$, $a_{10}=\frac{-7a_8}{3\cdot 9}=\frac{3\cdot 5\cdot 7a_0}{3^6\cdot 5\cdot 7\cdot 9}$.
The solution is

$$\begin{aligned}
y(x) &= a_0+a_1x+\frac{a_0}{3}x^2-\frac{a_0}{3^2\cdot 3}x^4+\frac{a_0}{3^3\cdot 5}x^6-\frac{a_0}{3^4\cdot 7}x^8+\cdots \\
&= a_1x+a_0\left(1+\frac{x^2}{3}-\frac{x^4}{3^2\cdot 3}+\frac{x^6}{3^3\cdot 5}-\frac{x^8}{3^4\cdot 7}+\cdots\right) \\
&= a_1x+a_0\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3^n(2n-1)}x^{2n}.
\end{aligned}$$

(b) Since the only singularities are $\pm\sqrt{3}i$, the radius of convergence is $\sqrt{3}$.

12 **3.** Find the Fourier series for the function

$$f(x) = \begin{cases} x-3, & -3 < x \le 0 \\ x+3, & 0 < x < 3, \end{cases} \qquad f(x+6) = f(x).$$

Simplify coefficients as much as possible. Draw a graph on the interval $-12 < x < 12$ of the function to which the Fourier series converges.

(a) Except for its value at $x = 0$, the function is odd, and therefore it has a Fourier sine series

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3},$$

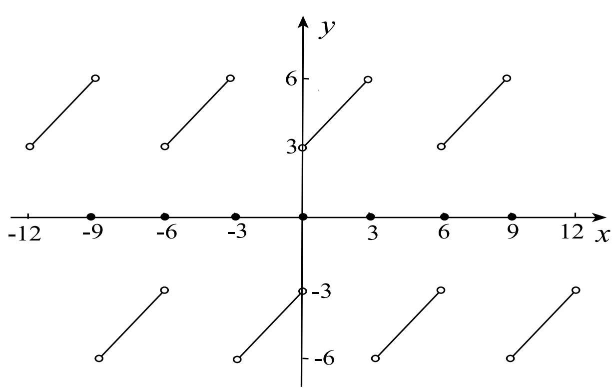
where

$$\begin{aligned} b_n &= \frac{2}{3} \int_0^3 (x+3) \sin \frac{n\pi x}{3}\, dx \\ &= \frac{2}{3} \left[\left\{ -\frac{3}{n\pi}(x+3) \cos \frac{n\pi x}{3} \right\}_0^3 - \int_0^3 -\frac{3}{n\pi} \cos \frac{n\pi x}{3}\, dx \right] \\ &= \frac{2}{3} \left[-\frac{18}{n\pi} \cos n\pi + \frac{9}{n\pi} + \frac{9}{n^2\pi^2} \left\{ \sin \frac{n\pi x}{3} \right\}_0^3 \right] \\ &= \frac{6}{n\pi}[1 + 2(-1)^{n+1}]. \end{aligned}$$

Thus,

$$\frac{f(x+) + f(x-)}{2} = \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{1 + 2(-1)^{n+1}}{n} \sin \frac{n\pi x}{3}.$$

(b)



12 **4.** (a) Find all eigenvalues and eigenfunctions for the Sturm-Liouville system

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + \lambda y = 0, \qquad 0 < x < 6,$$
$$y(0) = 0, \quad y(6) = 0.$$

You may assume that eigenvalues are larger than 4.

(b) Derive the weight function for the system?

(a) The auxiliary equation is $m^2 + 4m + \lambda = 0$ with solutions

$$m = \frac{-4 \pm \sqrt{16 - 4\lambda}}{2} = -2 \pm \sqrt{4 - \lambda}.$$

Since we may assume that $\lambda > 4$,

$$m = -2 \pm \sqrt{\lambda - 4}i.$$

If we set $\omega = \sqrt{\lambda - 4}$, then $m = -2 \pm \omega i$, and

$$y(x) = e^{-2x}(C_1 \cos \omega x + C_2 \sin \omega x).$$

The boundary conditions require

$$0 = y(0) = C_1, \qquad 0 = y(6) = e^{-12}(C_1 \cos 6\omega + C_2 \sin 6\omega).$$

The second of these requires

$$6\omega = n\pi \quad \Longrightarrow \quad 6\sqrt{\lambda_n - 4} = n\pi \quad \Longrightarrow \quad \lambda_n = \frac{n^2\pi^2}{36} + 4,$$

where $n \geq 1$ is an integer. Eigenfunctions are

$$y_n(x) = e^{-2x} \sin \frac{n\pi x}{6}.$$

(b) An integrating factor is $e^{\int 4\,dx} = e^{4x}$, so that

$$e^{4x}\frac{d^2y}{dx^2} + 4e^{4x}\frac{dy}{dx} + \lambda e^{4x} y = 0,$$

or,

$$\frac{d}{dx}\left[e^{4x}\frac{dy}{dx}\right] + \lambda e^{4x} y = 0.$$

This shows that the weight function is e^{4x}.