Math 3132 **Midterm Examination #2** **60 Minutes**

4 **1.** (a) Find all singular points for the differential equation

$$\sin 2x \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 3y = 0.$$

(b) Pick any one singular point from part (a) (your choice which one), and classify it as regular or irregular singular. Justify all statements.

Consider the functions

$$\frac{x}{\sin 2x}, \qquad \frac{3}{\sin 2x}.$$

Since the second fails to have Taylor series about $x = n\pi/2$, where n is an integer, these are singular points. The first function adds no further singular points.

To consider the singular point $x = 0$, we form

$$\frac{x^2}{\sin 2x} = \frac{x}{\sin 2x}, \qquad \frac{3x^2}{\sin 2x}.$$

Since both functions have limits as x approaches 0, they have Maclaurin series, and therefore $x = 0$ is regular singular.

11 **2** (a) Suppose that a Frobenius solution $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ is assumed for the differential equation

$$(x^2 - x)\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 0.$$

Find values for the indicial root r.

(b) Using the larger value of r in part (a), find a recurrence relation for the coefficients a_n. Do **NOT** iterate this relation.

(c) If you were to iterate the recurrence relation in part (b) (but don't do it), would you expect to get a general solution of the differential equation? Explain.

(a) When we substitute the series into the differential equation,

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} -(n+r)(n+r-1)a_n x^{n+r-1} \\
&\quad + \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} -2a_n x^{n+r} \\
&= \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=-1}^{\infty} -(n+r+1)(n+r)a_{n+1} x^{n+r} \\
&\quad + \sum_{n=-1}^{\infty} 3(n+r+1)a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} -2a_n x^{n+r} \\
&= [-r(r-1)a_0 x^{r-1} + 3ra_0 x^{r-1}] \\
&\quad + \sum_{n=0}^{\infty} [(n+r)(n+r-1)a_n - (n+r+1)(n+r)a_{n+1} + 3(n+r+1)a_{n+1} - 2a_n]x^{n+r}.
\end{aligned}$$

The indicial equation is $0 = -r(r-1) + 3r = -r^2 + 4r = r(-r+4)$, with solutions $r = 0$ and $r = 4$.

(b) With $r = 4$, the recurrence relation is

$$(n+4)(n+3)a_n - (n+5)(n+4)a_{n+1} + 3(n+5)a_{n+1} - 2a_n = 0,$$

for $n \geq 0$. Thus,

$$a_{n+1} = \frac{-2 + (n+4)(n+3)}{(n+5)(n+4) - 3(n+5)}a_n = \frac{n^2 + 7n + 10}{(n+5)(n+1)}a_n = \frac{(n+5)(n+2)}{(n+5)(n+1)}a_n = \frac{n+2}{n+1}a_n.$$

(c) No. The larger indicial root never gives a general solution.

10 **3.** (a) When a Frobenius solution $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ is substituted into the differential equation

$$2x(x+1)\frac{d^2y}{dx^2} + 3(x+1)\frac{dy}{dx} - y = 0,$$

the indicial roots are $r = 0$ and $= -1/2$. The recurrence relation corresponding to $r = 0$ is

$$a_{n+1} = -\frac{2n-1}{2n+3}a_n, \quad n \geq 0.$$

Iterate this relation to determine coefficients a_n. Write the corresponding solution of the differential equation in sigma notation, simplified as much as possible.

(b) Is your solution in part (a) a general solution? Would you have expected it to be? Explain.

(a) For $n = 0$, $a_1 = \frac{1}{3}a_0$.

For $n = 1$, $a_2 = -\frac{1}{5}a_1 = -\frac{1}{3 \cdot 5}a_0$.

For $n = 2$, $a_3 = -\frac{3}{7}a_2 = \frac{3}{3 \cdot 5 \cdot 7}a_0$.

For $n = 3$, $a_4 = -\frac{5}{9}a_3 = -\frac{3 \cdot 5}{3 \cdot 5 \cdot 7 \cdot 9}a_0$.

The solution is

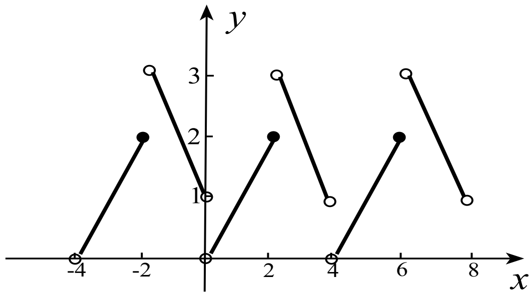
$$\begin{aligned} y(x) &= x^0 \left[a_0 + \frac{a_0}{3}x - \frac{a_0}{3 \cdot 5}x^2 + \frac{a_0}{5 \cdot 7}x^3 - \frac{a_0}{7 \cdot 9}x^4 + \cdots\right] \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)}x^n. \end{aligned}$$

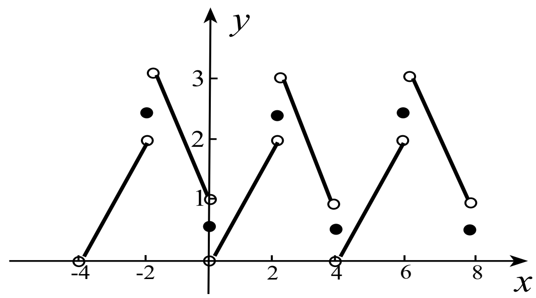
(b) The solution is not general. Theory says that when the indicial roots are different and do not differ by an integer, each indicial root gives a solution, and these must be superposed for a general solution.

15 **4.** (a) Suppose the function

$$f(x) = \begin{cases} x, & 0 < x \le 2 \\ 5 - x, & 2 < x < 4, \end{cases} \qquad f(x+4) = f(x),$$

is expanded in a Fourier series. On the left set of axes below, draw a graph of $f(x)$. On the right set of axes, draw a graph of the function to which the Fourier series converges.





(b) Find the coefficients a_n in the Fourier series of $f(x)$, simplified as much as possible. Do **NOT** calculate the coefficients b_n.

(b) $a_0 = \dfrac{1}{2}\displaystyle\int_0^4 f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx + \frac{1}{2}\int_2^4 (5-x)\,dx = \frac{1}{2}\left\{\frac{x^2}{2}\right\}_0^2 + \frac{1}{2}\left\{5x - \frac{x^2}{2}\right\}_2^4 = 3.$

For $n \ge 1$,

$$\begin{aligned}
a_n &= \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_2^4 (5-x)\cos\frac{n\pi x}{2}\,dx \\
&= \frac{1}{2}\left[\left\{\frac{2x}{n\pi}\sin\frac{n\pi x}{2}\right\}_0^2 - \int_0^2 \frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx\right] + \frac{1}{2}\left[\left\{\frac{2(5-x)}{n\pi}\sin\frac{n\pi x}{2}\right\}_2^4 - \int_2^4 -\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx\right] \\
&= -\frac{1}{n\pi}\left\{-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right\}_0^2 + \frac{1}{n\pi}\left\{-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right\}_2^4 \\
&= \frac{2}{n^2\pi^2}[\cos n\pi - 1] - \frac{2}{n^2\pi^2}[\cos 2n\pi - \cos n\pi] \\
&= \frac{2}{n^2\pi^2}[(-1)^n - 1 - 1 + (-1)^n] = -\frac{4}{n^2\pi^2}[1 + (-1)^{n+1}].
\end{aligned}$$