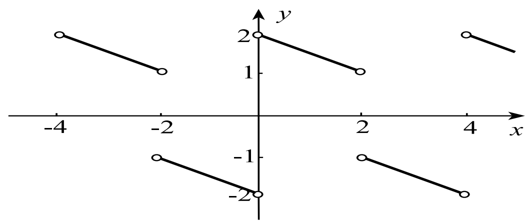
Midterm Examination #3 Solutions

1. (a) Find the Fourier series for the function in the figure below (simplified as much as possible).
(b) Draw a graph of the function to which the Fourier series in part (a) converges.



(a) Since the function is odd, its Fourier series will be a sine series

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2},$$

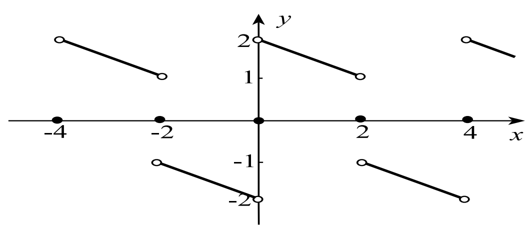
where

$$b_n = \frac{2}{2}\int_0^2 \frac{1}{2}(4-x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\left[\left\{-\frac{2}{n\pi}(4-x)\cos\frac{n\pi x}{2}\right\}_0^2 - \int_0^2 \frac{2}{n\pi}\cos\frac{n\pi x}{2}\,dx\right]$$
$$= \frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi + \frac{8}{n\pi} - \frac{2}{n\pi}\left\{\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right\}_0^2\right] = \frac{2}{n\pi}[2+(-1)^{n+1}].$$

The Fourier sine series is therefore

$$\sum_{n=1}^{\infty}\frac{2}{n\pi}[2+(-1)^{n+1}]\sin\frac{n\pi x}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2+(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}.$$

(b)



2. A general solution of the differential equation

$$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + \lambda x^2 y = 0$$

is known to be

$$y(x) = \begin{cases} \dfrac{C_1}{x} + C_2, & \text{if } \lambda = 0 \\ \dfrac{C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x}{x}, & \text{if } \lambda > 0. \end{cases}$$

Do **NOT** show this. Use it to find eigenvalues and eigenfunctions of the Sturm-Liouville system

$$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + \lambda x^2 y = 0, \quad 0 < x < 1,$$

$$y(1) = 0.$$

Instead of a boundary condition at $x = 0$, impose the condition that

$$\lim_{x\to 0^+} y(x) \text{ must exist.}$$

Assume that all eigenvalues of the system are nonnegative and consider the two cases $\lambda = 0$ and $\lambda > 0$.

Consider first the case that $\lambda = 0$. If the limit $\lim_{x\to 0^+} y(x)$ is to exist, then C_1 must be set equal to zero. The condition $y(1) = 0$ then implies that $0 = C_2$. Thus, $\lambda = 0$ cannot be an eigenvalue. Consider now the case that $\lambda > 0$. If the limit $\lim_{x\to 0^+} y(x)$ is to exist, then C_1 must again be set equal to zero. The condition $y(1) = 0$ then implies that

$$C_2 \sin\sqrt{\lambda} = 0 \quad \Longrightarrow \quad \sqrt{\lambda} = n\pi, \quad n \neq 0 \text{ an integer.}$$

Eigenvalues are therefore $\lambda_n = n^2\pi^2$, where $n \geq 1$. Corresponding eigenfunctions are $y_n(x) = (C_2/x)\sin n\pi x$.

3. Find the eigenfunction expansion of the function $f(x) = 3x$, $0 < x < L$, in terms of the eigenfunctions of the Sturm-Liouville system

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < L,$$
$$y'(0) = 0,$$
$$y(L) = 0.$$

Simplify coefficients as much as possible.

From the table, eigenfunctions are $y_n(x) = \cos\dfrac{(2n-1)\pi x}{2L}$. The eigenfunction expansion is

$$3x = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2L},$$

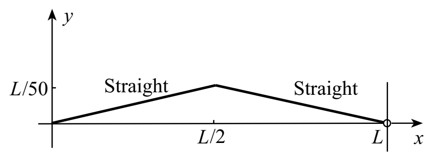
where

$$\begin{aligned}
c_n &= \frac{2}{L}\int_0^L 3x\cos\frac{(2n-1)\pi x}{2L}\,dx \\
&= \frac{6}{L}\left[\left\{\frac{2Lx}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\right\}_0^L - \int_0^L \frac{2L}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= \frac{6}{L}\left[\frac{2L^2}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2} + \left\{\frac{4L^2}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi x}{2L}\right\}_0^L\right] \\
&= \frac{6}{L}\left[\frac{2L^2(-1)^{n+1}}{(2n-1)\pi} - \frac{4L^2}{(2n-1)^2\pi^2}\right] \\
&= \frac{12L}{\pi}\left[\frac{(-1)^{n+1}}{2n-1} - \frac{2}{(2n-1)^2\pi}\right].
\end{aligned}$$

The eigenfunction expansion is

$$3x = \frac{12L}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{2n-1} - \frac{2}{(2n-1)^2\pi}\right]\cos\frac{(2n-1)\pi x}{2L}.$$

4. A string with linear density ρ is stretched tightly between the points $x = 0$ and $x = L$ on the x-axis. The tension in the string is a constant τ. The displacement of the string at time $t = 0$ is shown in the figure below, and from this position, it is released. The left end of the string is fixed on the x-axis, but the right end is looped around a vertical rod, and can move vertically without friction. If gravity is taken into account, as well as a damping force proportional to velocity, what is the initial-value problem for displacement $y(x,t)$ of the string? Include the partial differential equation, and all boundary and initial conditions, and include intervals on which they must be satisfied.



The initial boundary vavlue problem is

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2} - g - \frac{\beta}{\rho}\frac{\partial y}{\partial t}, \quad 0 < x < L, \quad t > 0,\\
y(0,t) &= 0, \quad t > 0,\\
y_x(L,t) &= 0, \quad t > 0,\\
y(x,0) &= f(x), \quad 0 < x < L,\\
y_t(x,0) &= 0, \quad 0 < x < L,
\end{aligned}$$

where $c^2 = \tau/\rho$, $g = 9.81$, and $f(x) = \begin{cases} x/25, & 0 < x \le L/2 \\ (L-x)/25, & L/2 < x < L. \end{cases}$

5. For what value of k are the functions x and $1 - 2kx^2$ orthogonal on the interval $0 \le x \le 1$ with respect to the weight function $w(x) = x$?

For orthogonality we must have

$$0 = \int_0^1 x(x)(1 - 2kx^2)\,dx = \int_0^1 (x^2 - 2kx^4)\,dx = \left\{\frac{x^3}{3} - \frac{2kx^5}{5}\right\}_0^1 = \frac{1}{3} - \frac{2k}{5}.$$

Thus, $k = 5/6$.