Midterm Examination #3 MATH3132 Mathematical Methods for Engineers 3

14 **1.** (a) Find the Fourier series for the function

$$f(x) = 3|x|, \quad -2 \le x \le 2, \qquad f(x+4) = f(x).$$

Simplify the series as much as possible.

(b) Use the Fourier series in part (a) to find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

(a) Since the function is even, its Fourier series will be a Fourier cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2},$$

where $a_n = \dfrac{2}{2}\displaystyle\int_0^2 3x \cos\frac{n\pi x}{2} dx$. We find

$$a_0 = 3\int_0^2 x\,dx = 3\left\{\frac{x^2}{2}\right\}_0^2 = 6,$$

and for $n \ge 1$,

$$\begin{aligned} a_n &= 3\int_0^2 x\cos\frac{n\pi x}{2}dx = 3\left\{\frac{2x}{n\pi}\sin\frac{n\pi x}{2}\right\}_0^2 - 3\int_0^2 \frac{2}{n\pi}\sin\frac{n\pi x}{2}dx \\ &= -\frac{6}{n\pi}\left\{\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right\}_0^2 = \frac{12}{n^2\pi^2}[(-1)^n - 1]. \end{aligned}$$

The Fourier cosine series is

$$3 + \sum_{n=1}^{\infty} \frac{12}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi x}{2} = 3 - \frac{24}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2}.$$

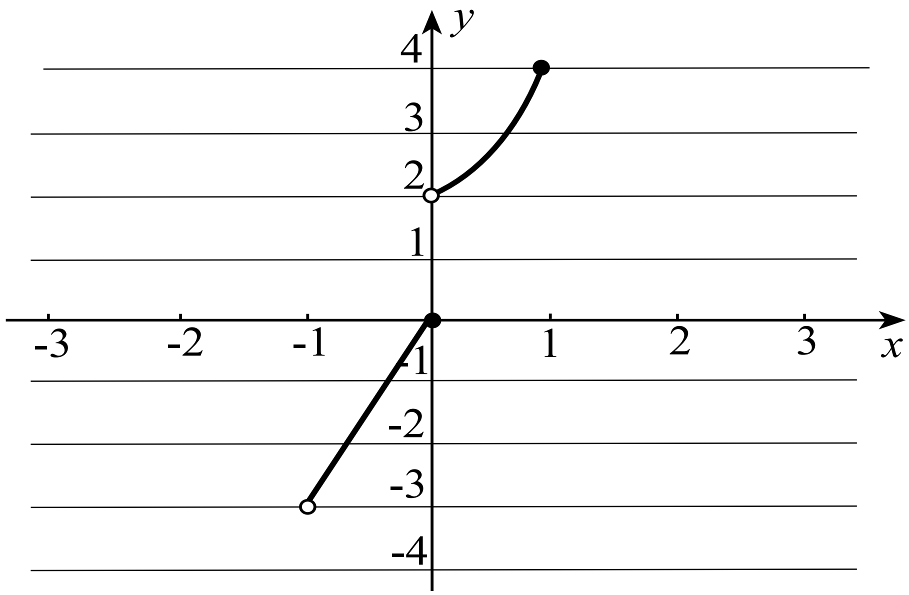
(b) Since the series converges to 0 at $x = 0$, we can write that

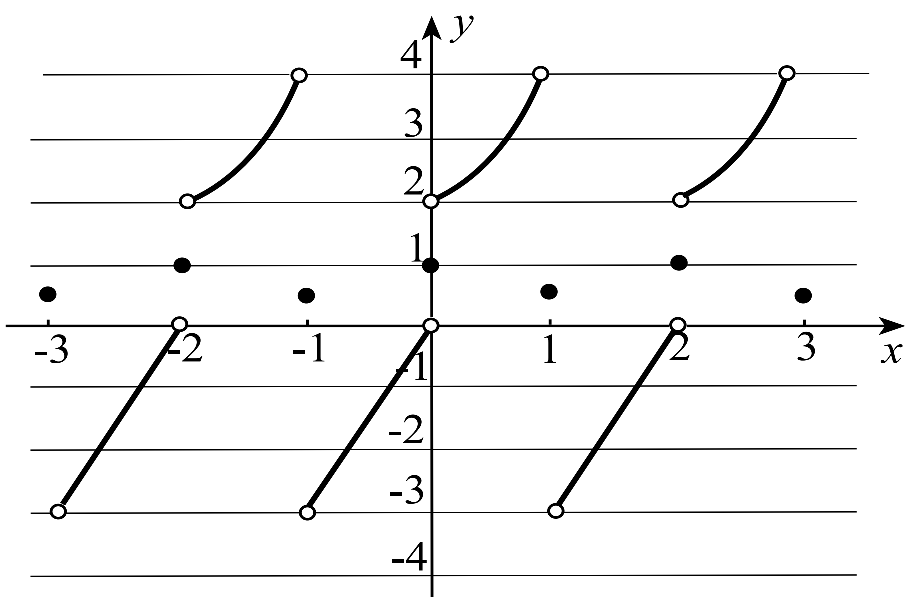
$$0 = 3 - \frac{24}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \quad\Longrightarrow\quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{3\pi^2}{24} = \frac{\pi^2}{8}.$$

6 **2.** The function in the first graph below is to be extended as a periodic function with period 2. Suppose that the Fourier series of this extension is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x).$$

On the second set of axes, draw a graph of the function to which the Fourier series converges on the interval $-3 \leq x \leq 3$.





6 **3.** The function $f(x) = xe^{-3x}$, $0 \leq x \leq 4$, is to be expressed in terms of the eigenfunctions of the Sturm-Liouville system

$$\begin{aligned} \frac{d^2y}{dx^2} + \lambda y &= 0, \quad 0 < x < 4, \\ y'(0) &= 0, \\ y(4) &= 0. \end{aligned}$$

(a) What is the form of the eigenfunction expansion? You may use the table on page 1 to determine eigenfunctions of the system.

(b) What integral defines the coefficients in the eigenfunction expansion? **Do NOT evaluate the integral.**

(a) Eigenfunctions of the Sturm-Liouville system are $y_n(x) = \cos \dfrac{(2n-1)\pi x}{8}$. The eigenfunction expansion of $f(x)$ is

$$f(x) = xe^{-3x} = \sum_{n=1}^{\infty} C_n \cos \frac{(2n-1)\pi x}{8}.$$

(b) Coefficients are given by

$$C_n = \frac{2}{4} \int_0^4 xe^{-3x} \cos \frac{(2n-1)\pi x}{8} dx.$$

14 **4.** (a) Find eigenvalues and eigenfunctions of the Sturm-Liouville system

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + \lambda y = 0, \quad 0 < x < 1,$$
$$y(0) = 0,$$
$$y(1) = 0.$$

You may assume that eigenvalues are greater than 1.

(b) What is the weight function for the system?

(a) The auxiliary equation is

$$m^2 - 2m + \lambda = 0 \quad \Longrightarrow \quad m = \frac{2 \pm \sqrt{4 - 4\lambda}}{2} = 1 \pm \sqrt{1 - \lambda}.$$

Since $\lambda > 1$, we write that

$$m = 1 \pm \sqrt{\lambda - 1}i = 1 \pm \omega i, \qquad \text{where } \omega = \sqrt{\lambda - 1}.$$

The general solution of the differential equation is

$$y(x) = e^x(C_1 \cos \omega x + C_2 \sin \omega x).$$

For $y(0) = 0$,

$$0 = e^0(C_1) = C_1.$$

For $y(1) = 0$,

$$0 = e^1(C_1 \cos \omega + C_2 \sin \omega) = C_2 e \sin \omega.$$

Since we cannot set $C_2 = 0$, we must set

$$\sin \omega = 0 \quad \Longrightarrow \omega = n\pi, \quad n \text{ an integer.}$$

Thus, $n\pi = \sqrt{\lambda - 1}$, from which $\lambda_n = 1 + n^2\pi^2$, $n \geq 1$, are the eigenvalues. Eigenfunctions are

$$y_n(x) = C_2 e^x \sin n\pi x.$$

(b) An integrating factor is $e^{\int -2dx} = e^{-2x}$. When we multiply each term in the differential equation by this factor, we obtain

$$e^{-2x}\frac{d^2y}{dx^2} - 2e^{-2x}\frac{dy}{dx} + \lambda y e^{-2x} = 0,$$

or,

$$\frac{d}{dx}\left(e^{-2x}\frac{dy}{dx}\right) + \lambda e^{-2x} y = 0.$$

This shows that the weight function is e^{-2x}.