**EXERCISES 17.3**

**2.** The only singular point of the differential equation is $x = 0$. Since $x\left(\dfrac{2}{x^2}\right) = \dfrac{2}{x}$ does not have a Maclaurin series, $x = 0$ is an irregular singular point.

**4.** The only singular point of the differential equation is $x = 0$. Since $x^2\left(\dfrac{1}{x^3}\right) = \dfrac{1}{x}$ does not have a Maclaurin series, $x = 0$ is an irregular singular point.

**6.** Singular points of the differential equation are $x = \pm 1$. Since

$$(x-1)\left(\frac{-1}{x^2-1}\right) = \frac{-1}{x+1} \qquad \text{and} \qquad (x-1)^2\left(\frac{x}{x^2-1}\right) = \frac{x(x-1)}{x+1},$$

both have convergent Taylor series around $x = 1$, $x = 1$ is a regular singular point. Similar reasoning shows that $x = -1$ is also regular singular.

**8.** Singular points of the differential equation are $x = (2n+1)\pi/2$, where $n$ is an integer. The function $[x - (2n+1)\pi/2]\sin x$ certainly has a Taylor series about $x = (2n+1)\pi/2$. Consider now the function $[x-(2n+1)\pi/2]^2(-3\tan x)$. We use L'Hôpital's rule to show that the function has a limit as $x$ approaches $(2n+1)\pi/2$,

$$\lim_{x\to(2n+1)\pi/2}[x-(2n+1)\pi/2]^2(-3\tan x) = \lim_{x\to(2n+1)\pi/2}\frac{-3[x-(2n+1)\pi/2]^2}{\cot x}$$
$$= \lim_{x\to(2n+1)\pi/2}\frac{-6[x-(2n+1)\pi/2]}{-\csc^2 x} = 0.$$

It now follows that the function $[x-(2n+1)\pi/2]^2(-3\tan x)$ has a Taylor series about $x = (2n+1)\pi/2$, and the $x = (2n+1)\pi/2$ are regular singular points.

**10.** This differential equation has no real singular points.

**12.** To find Frobenius solutions about $x = 0$, we substitute $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ into the differential equation

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+2} + \sum_{n=0}^{\infty}-(a_n/4)x^{n+r}\\
&= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=2}^{\infty}a_{n-2} x^{n+r} + \sum_{n=0}^{\infty}-(a_n/4)x^{n+r}\\
&= \left[r(r-1)a_0 + ra_0 - \frac{a_0}{4}\right]x^r + \left[(r+1)ra_1 + (r+1)a_1 - \frac{a_1}{4}\right]x^{r+1}\\
&\quad + \sum_{n=2}^{\infty}\left[(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2} - \frac{a_n}{4}\right]x^{n+r}.
\end{aligned}$$

The only way the series can vanish for all $x$ in some interval around $x = 0$ is for each and every coefficient to vanish. The agreement is that the coefficient of the lowest power of $x$ will always be used to determine $r$; that is, the indicial equation is

$$0 = r(r-1) + r - \frac{1}{4} = r^2 - \frac{1}{4} \quad \Longrightarrow \quad r = \pm\frac{1}{2},$$

a case 3 situation. The coefficient of $x^{r+1}$ requires

$$[(r+1)r + (r+1) - 1/4]a_1 = 0.$$

Since this vanishes when $r = -1/2$, there is the possibility that this indicial root may give a general solution. When $r = -1/2$, equating the remainder of the coefficients to zero gives

$$\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)a_n + \left(n-\frac{1}{2}\right)a_n + a_{n-2} - \frac{a_n}{4} = 0.$$

This gives the recurrence relation

$$a_n = \frac{-a_{n-2}}{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)+\left(n-\frac{1}{2}\right)-\frac{1}{4}} = \frac{-4a_{n-2}}{(2n-1)(2n-3)+2(2n-1)-1} = \frac{-a_{n-2}}{n(n-1)}, \quad n \geq 2.$$

Iteration of this relation yields

$$a_2 = \frac{-a_0}{2}, \quad a_3 = \frac{-a_1}{6}, \quad a_4 = \frac{-a_2}{12} = \frac{a_0}{4!}, \quad a_5 = \frac{-a_3}{20} = \frac{a_1}{5!}, \ldots .$$

Thus, when $r = -1/2$, we obtain a general solution

$$\begin{aligned} y(x) &= x^{-1/2}\left[a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\right] \\ &= \frac{a_0 \cos x + a_1 \sin x}{\sqrt{x}}, \end{aligned}$$

valid for all $x$, except the singular point $x = 0$.

**14.** To find Frobenius solutions about $x = 0$, we substitute $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ into the differential equation

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} -(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} -a_n x^{n+r} \\ &= \sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} + \sum_{n=-1}^{\infty}(n+r+1)a_{n+1}x^{n+r} + \sum_{n=0}^{\infty} -(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} -a_n x^{n+r} \\ &= [r(r-1)a_0 + ra_0]x^{r-1} + \sum_{n=0}^{\infty}\left[(n+r+1)(n+r)a_{n+1} + (n+r+1)a_{n+1} - (n+r)a_n - a_n\right]x^{n+r}. \end{aligned}$$

The only way the series can vanish for all $x$ in some interval around $x = 0$ is for each and every coefficient to vanish. The agreement is that the coefficient of the lowest power of $x$ will always be used to determine $r$; that is, the indicial equation is

$$0 = r(r-1) + r = r^2 \quad \Longrightarrow \quad r = 0,$$

a case 2 situation. When we set $r = 0$ and equate the remainder of the coefficients to zero,

$$(n+1)na_{n+1} + (n+1)a_{n+1} - na_n - a_n = 0 \qquad \Longrightarrow \qquad a_{n+1} = \frac{a_n}{n+1}, \quad n \geq 0.$$

Iteration of this relation yields

$$a_1 = a_0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \quad a_4 = \frac{a_3}{4} = \frac{a_0}{4!}, \ldots .$$

Thus, when $r = 0$, we obtain the solution

$$y(x) = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = a_0 e^x.$$

**16.** To find Frobenius solutions about $x = 0$, we substitute $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ into the differential equation

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= \sum_{n=0}^{\infty}[(n+r)(n+r-1)a_n + 3(n+r)a_n + a_n]x^{n+r}. \end{aligned}$$

The only way the series can vanish for all $x$ in some interval around $x = 0$ is for each and every coefficient to vanish. The agreement is that the coefficient of the lowest power of $x$ will always be used to determine $r$; that is, the indicial equation is

$$0 = r(r-1) + 3r + 1 = (r+1)^2 \quad \Longrightarrow \quad r = -1, -1,$$

a case 2 situation. When we set $r = -1$ and equate the remainder of the coefficients to zero,

$$[(n-1)(n-2) + 3(n-1) + 1]a_n = 0 \qquad \Longrightarrow \qquad n^2 a_n = 0, \quad n \geq 1.$$

This requires that $a_n = 0$ for $n \geq 1$. Thus, when $r = -1$, we obtain the solution

$$y(x) = x^{-1} a_0 = \frac{a_0}{x}.$$

**18.** To find Frobenius solutions about $x = 0$, we substitute $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ into the differential equation

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 5(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} -10(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} -5a_n x^{n+r} \\
&= \sum_{n=-1}^{\infty} 2(n+r+1)(n+r)a_{n+1} x^{n+r} + \sum_{n=-1}^{\infty} 5(n+r+1)a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} -10(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} -5a_n x^{n+r} \\
&= [2r(r-1)a_0 + 5ra_0]x^{r-1} + \sum_{n=0}^{\infty} [2(n+r+1)(n+r)a_{n+1} + 5(n+r+1)a_{n+1} - 10(n+r)a_n - 5a_n]\, x^{n+r}.
\end{aligned}$$

The only way the series can vanish for all $x$ in some interval around $x = 0$ is for each and every coefficient to vanish. The agreement is that the coefficient of the lowest power of $x$ will always be used to determine $r$; that is, the indicial equation is

$$0 = 2r(r-1) + 5r = r(2r+3) \quad \Longrightarrow \quad r = 0, -3/2,$$

a case 1 situation. When we set $r = 0$ and equate the remainder of the coefficients to zero,

$$2(n+1)na_{n+1} + 5(n+1)a_{n+1} - 10na_n - 5a_n = 0,$$

from which

$$a_{n+1} = \frac{(10n+5)a_n}{2n(n+1) + 5(n+1)} = \frac{5(2n+1)a_n}{(2n+5)(n+1)}, \quad n \geq 0.$$

Iteration of this relation yields

$$\begin{aligned}
a_1 &= \frac{5a_0}{5} = a_0, \quad a_2 = \frac{5 \cdot 3a_1}{7 \cdot 2} = \frac{5 \cdot 3a_0}{7 \cdot 2}, \quad a_3 = \frac{5 \cdot 5a_2}{9 \cdot 3} = \frac{5^3 \cdot 3a_0}{2 \cdot 3 \cdot 7 \cdot 9}, \\
a_4 &= \frac{5 \cdot 7a_3}{11 \cdot 4} = \frac{5^4 \cdot 3a_0}{2 \cdot 3 \cdot 4 \cdot 9 \cdot 11}, \quad a_5 = \frac{5 \cdot 9a_4}{13 \cdot 5} = \frac{5^5 \cdot 3}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 11 \cdot 13} a_0, \ldots .
\end{aligned}$$

If we set $a_0 = 1$, a solution corresponding to $r = 0$ is

$$\begin{aligned}
y_1(x) &= x^0 \left(1 + x + \frac{3 \cdot 5}{2 \cdot 7}x^2 + \frac{5^3 \cdot 3}{2 \cdot 3 \cdot 7 \cdot 9}x^3 + \frac{5^4 \cdot 3}{2 \cdot 3 \cdot 4 \cdot 9 \cdot 11}x^4 + \frac{5^5 \cdot 3}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 11 \cdot 13}x^5 + \cdots\right) \\
&= 3 \sum_{n=0}^{\infty} \frac{5^n}{n!(2n+1)(2n+3)} x^n.
\end{aligned}$$

Now for the indicial root $r = -3/2$. When we set $r = -3/2$ in

$$2(n+r+1)(n+r)a_{n+1} + 5(n+r+1)a_{n+1} - 10(n+r)a_n - 5a_n = 0,$$

we obtain

$$2\left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)a_{n+1} + 5\left(n - \frac{1}{2}\right)a_{n+1} - 10\left(n - \frac{3}{2}\right)a_n - 5a_n = 0,$$

or,

$$(2n-1)(2n-3)a_{n+1} + 5(2n-1)a_{n+1} - (20n-30)a_n - 10a_n = 0.$$

This gives the recurrence relation

$$a_{n+1} = \frac{20n-30+10}{(2n-1)(2n-3)+5(2n-1)}a_n = \frac{10(n-1)a_n}{(2n-1)(n+1)}, \quad n \geq 0.$$

Iteration yields $a_1 = 10a_0$, and $0 = a_2 = a_3 = \ldots$. If we set $a_0 = 1$, a solution corresponding to the indicial root $r = -3/2$ is

$$y_2(x) = x^{-3/2}(1+10x) = x^{-3/2} + 10x^{-1/2}.$$

A general solution of the differential equation is $y(x) = Ay_1(x) + By_2(x)$.

**20.** To find Frobenius solutions about $x = 0$, we substitute $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ into the differential equation

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} -(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} -2a_n x^{n+r} \\
&= \sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=-1}^{\infty} -(n+r+1)a_{n+1}x^{n+r} + \sum_{n=0}^{\infty} -2a_n x^{n+r} \\
&= [r(r-1)a_0 - ra_0]x^{r-1} + \sum_{n=0}^{\infty}\left[(n+r+1)(n+r)a_{n+1} + (n+r)a_n - (n+r+1)a_{n+1} - 2a_n\right]x^{n+r}.
\end{aligned}$$

The only way the series can vanish for all $x$ in some interval around $x = 0$ is for each and every coefficient to vanish. The agreement is that the coefficient of the lowest power of $x$ will always be used to determine $r$; that is, the indicial equation is

$$0 = r(r-1) - r = r(r-2) \quad \Longrightarrow \quad r = 0, 2,$$

a case 3 situation. When we set $r = 0$ and equate the remainder of the coefficients to zero,

$$(n+1)na_{n+1} + na_n - (n+1)a_{n+1} - 2a_n = 0 \quad \Longrightarrow \quad a_{n+1} = \frac{(2-n)a_n}{n(n+1)-(n+1)} = \frac{(2-n)a_n}{(n+1)(n-1)}, \quad n \geq 0.$$

When we set $n = 0$ in this recurrence relation, we obtain $a_1 = -2a_0$. To substitute $n = 1$, we must return to the version of the recurrence relation without division,

$$2(0)a_2 = (1)a_1 \Longrightarrow \quad a_1 = 0.$$

This, in turn, implies that all coefficients are zero except $a_2$ which is arbitrary. In other words, the solution corresponding to $r = 0$ is $y(x) = a_2x^2$. When we work with $r = 2$, we obtain

$$(n+3)(n+2)a_{n+1} + (n+2)a_n - (n+3)a_{n+1} - 2a_n = 0 \quad \Longrightarrow \quad a_{n+1} = \frac{-na_n}{(n+3)(n+1)}, \quad n \geq 0.$$

This relation implies that all coefficients, except $a_0$ vanish. The solution corresponding to $r = 2$ is therefore $y(x) = a_0x^2$.

**22.** We solved this differential equation earlier in this section. Substitution of $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ led to

$$0 = x^r\left[(r^2+3r+2)a_0 + (r^2+5r+6)a_1x + \sum_{n=2}^{\infty}\{[(n+r)(n+r+3)+2]a_n + a_{n-2}\}x^n\right].$$

By using $r^2 + 3r + 2 = 0$ as the indicial equation, we obtained $r = -1$ and $r = -2$, and the solution

$$y(x) = \frac{a_0\cos x + a_1\sin x}{x^2}.$$

Suppose now that we choose to determine the indicial roots by setting

$$0 = r^2 + 5r + 6 = (r+2)(r+3) \quad \Longrightarrow \quad r = -2, -3.$$

With $r = -3$,

$$0 = (r^2 + 3r + 2)a_0 = (9 - 9 + 2)a_0 \quad \Longrightarrow \quad a_0 = 0.$$

The remaining coefficients imply that

$$(n-3)(n-4)a_n + 4(n-3)a_n + a_{n-2} + 2a_n = 0 \quad \Longrightarrow \quad a_n = \frac{-a_{n-2}}{(n-1)(n-2)}, \quad n \geq 2.$$

To set $n = 2$, we return to the nondivisional form of this relation,

$$(0)a_2 = -a_0 \quad \Longrightarrow \quad a_0 = 0.$$

Iteration of the recurrence relation now gives

$$a_3 = \frac{-a_1}{2}, \quad a_4 = \frac{-a_2}{3 \cdot 2}, \quad a_5 = \frac{-a_3}{4 \cdot 3} = \frac{a_1}{4!}, \quad a_6 = \frac{-a_4}{5 \cdot 4} = \frac{a_2}{5!}.$$

Thus, from the indicial root $r = -3$, we obtain a general solution

$$\begin{aligned} y(x) &= x^{-3}\left[a_1\left(x - \frac{x^3}{2!} + \frac{x^5}{4!} + \cdots\right) + a_2\left(x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \cdots\right)\right] \\ &= x^{-2}\left[a_1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + a_2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\right] \\ &= \frac{a_1 \cos x + a_2 \sin x}{x^2}. \end{aligned}$$

# CHAPTER 18

## EXERCISES 18.1

**2.** The Fourier coefficients for this function are

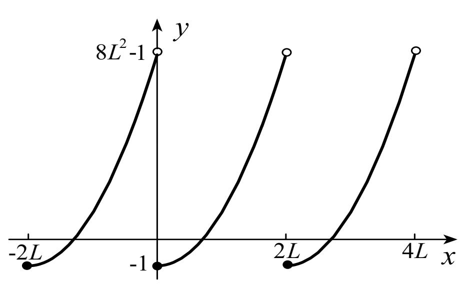
$$a_0 = \frac{1}{L}\int_0^{2L} (2x^2 - 1)\,dx = \frac{16L^2}{3} - 2,$$

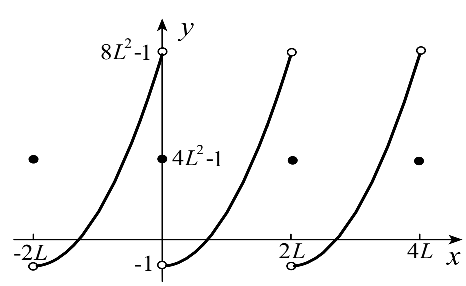
$$a_n = \frac{1}{L}\int_0^{2L} (2x^2 - 1)\cos\frac{n\pi x}{L}\,dx = \frac{8L^2}{n^2\pi^2}, \qquad b_n = \frac{1}{L}\int_0^{2L} (2x^2 - 1)\sin\frac{n\pi x}{L}\,dx = \frac{-8L^2}{n\pi}.$$

The Fourier series of $f(x)$ is therefore

$$\frac{8L^2}{3} - 1 + \sum_{n=1}^{\infty}\left(\frac{8L^2}{n^2\pi^2}\cos\frac{n\pi x}{L} - \frac{8L^2}{n\pi}\sin\frac{n\pi x}{L}\right) = \frac{8L^2}{3} - 1 + \frac{8L^2}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos\frac{n\pi x}{L} - \frac{\pi}{n}\sin\frac{n\pi x}{L}\right).$$

The function $f(x)$ and the function to which its Fourier series converges are shown below.




**4.** The Fourier coefficients for this function are

$$a_0 = \frac{1}{L}\int_0^{2L} 3x\,dx = 6L, \quad a_n = \frac{1}{L}\int_0^{2L} 3x\cos\frac{n\pi x}{L}\,dx = 0, \quad b_n = \frac{1}{L}\int_0^{2L} 3x\sin\frac{n\pi x}{L}\,dx = -\frac{6L}{n\pi}.$$