MATH 3700 Test 1 October 8, 2014 **60 minutes**

Student Name (Print): __________________ **Student Number:** __________

Values

7 **1.** Find the fourth roots of $-1+i$. Express answers in Cartesian form.

Finding the fourth roots is equivalent to solving the equation $z^4 = 1 - i$. If we set $z = re^{\theta i}$, then

$$r^4 e^{4\theta i} = \sqrt{2}e^{3\pi i/4}.$$

This implies that

$$r^4 = \sqrt{2} \qquad \text{and} \qquad 4\theta = \frac{3\pi}{4} + 2k\pi, \quad k \text{ an integer.}$$

Thus,

$$r = 2^{1/8} \qquad \text{and} \qquad \theta = \frac{3\pi}{16} + \frac{k\pi}{2}.$$

The fourth roots are therefore

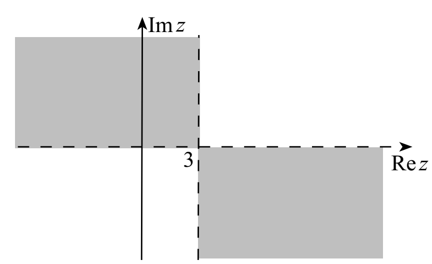
$$\begin{aligned} z = 2^{1/8}e^{(3\pi/16 + k\pi/2)i} &= 2^{1/8}\left[\cos\left(\frac{3\pi}{16} + \frac{k\pi}{2}\right) + i\sin\left(\frac{3\pi}{16} + \frac{k\pi}{2}\right)\right] \\ &= 2^{1/8}\cos\frac{(8k+3)\pi}{16} + i2^{1/8}\sin\frac{(8k+3)\pi}{16}, \quad k = 0, 1, 2, 3. \end{aligned}$$

8 **2.** (a) Draw the region in the z-plane described by the inequality

$$(\operatorname{Im} z)(\operatorname{Re} z - 3) < 0.$$

(b) State whether the region is (i) open, (ii) closed, (iii) connected, (iv) bounded, and (v) a domain.

(a) If we set $z = x + yi$, then $y(x-3) < 0$. This means that either $y < 0$ and $x > 3$, or $y > 0$ and $x < 3$. The region is shown to the right.
(b) The region is open, not closed, not connected, not bounded, and not a domain.



7 **3.** If $z = x + yi$, show that the bilinear function

$$w = f(z) = \frac{4}{z+3}$$

maps the straight line $3x - 4y + 9 = 0$ in the z-plane to a straight line in the w-plane. What is the equation of the image line in the w-plane?

If we write $w(z+3) = 4$, then $z = \frac{4}{w} - 3$. We now set $z = x + yi$ and $w = u + vi$,

$$x + yi = \frac{4}{u + vi} - 3 = \frac{4(u - vi)}{u^2 + v^2} - 3.$$

Thus,

$$x = \frac{4u}{u^2 + v^2} - 3, \qquad y = \frac{-4v}{u^2 + v^2}.$$

The image of $3x - 4y + 9 = 0$ is therefore

$$0 = 3\left(\frac{4u}{u^2 + v^2} - 3\right) - 4\left(\frac{-4v}{u^2 + v^2}\right) + 9 = \frac{12u}{u^2 + v^2} - 9 + \frac{16v}{u^2 + v^2} + 9.$$

If we multiply by $(u^2 + v^2)/4$, the image is $0 = 3u + 4v$, a straight line.

8 **4.** Determine whether the function

$$f(z) = \begin{cases} \dfrac{\text{Im}(z^3)}{z^3}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is continuous at $z = 0$. Justify your answer.

Since the function is defined at $z = 0$, we consider its limit as $z \to 0$, $\lim_{z \to 0} \frac{\text{Im}(z^3)}{z^3}$. If we approach $z = 0$ along the real axis by setting $z = x$, then

$$\lim_{z \to 0} \frac{\text{Im}(z^3)}{z^3} = \lim_{x \to 0} \frac{\text{Im}(x^3)}{x^3} = \lim_{x \to 0} \frac{0}{x^3} = 0.$$

If we approach $z = 0$ along the imaginary axis by setting $z = yi$, then

$$\lim_{z \to 0} \frac{\text{Im}(z^3)}{z^3} = \lim_{y \to 0} \frac{\text{Im}(yi)^3}{(yi)^3} = \lim_{y \to 0} \frac{(-y^3 i)}{-y^3 i} = \lim_{y \to 0} \frac{-y^3}{-y^3 i} = -i.$$

Since L depends on the mode of approach, the limit does not exist. The function is therefore discontinuous at $z = 0$.

5 **5.** You are given that the function

$$f(z) = \left(x^3 - 3xy^2 + \frac{3x}{x^2+y^2}\right) + \left(3x^2y - y^3 - \frac{3y}{x^2+y^2}\right) i$$

is analytic in some domain D. Find an expression for $f'(z)$ in D, but do not simplify your answer.

Since the function is analytic,

$$f'(z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i = \left[3x^2 - 3y^2 + \frac{(x^2+y^2)(3) - 3x(2x)}{(x^2+y^2)^2}\right] + \left[6xy - \frac{3y(-1)(2x)}{(x^2+y^2)^2}\right] i.$$

5 **6.** Prove that if $u(x, y)$ and $v(x, y)$ are harmonic conjugates in a domain D, then the product $u(x, y)v(x, y)$ is also harmonic in D.

Since u and v are harmonic conjugates, they have continuous second derivatives that satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

They also satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The function uv will have continuous second derivatives, and

$$\begin{aligned}
\frac{\partial^2 (uv)}{\partial x^2} + \frac{\partial^2 (uv)}{\partial y^2} &= \left(u\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + v\frac{\partial^2 u}{\partial x^2}\right) + \left(u\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + v\frac{\partial^2 u}{\partial y^2}\right) \\
&= u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) \\
&= 2\left[\frac{\partial u}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right] = 0.
\end{aligned}$$

Thus, uv is harmonic.