MATH 3700 Test 2 November 7, 2013 60 minutes

Student Name (Print): ____________________ **Student Number:** ____________

Values

4 **1.** Use definitions of the sine and cosine functions to prove that

$$\cos 2z = 1 - 2\sin^2 z.$$

$$1 - 2\sin^2 z = 1 - 2\left(\frac{e^{zi} - e^{-zi}}{2i}\right)^2 = 1 - \frac{2}{-4}(e^{2zi} - 2 + e^{-2zi})$$
$$= 1 + \frac{1}{2}(e^{2zi} - 2 + e^{-2zi}) = \frac{1}{2}(e^{2zi} + e^{-2zi}) = \cos 2z$$

5 **2.** Find all solutions of the equation

$$e^{2z} = 1 + \sqrt{3}i.$$

Express answers in Cartesian form.

If we take logarithms,

$$2z = \log(1 + \sqrt{3}i) = \ln|1 + \sqrt{3}i| + i\arg(1 + \sqrt{3}i).$$

Thus,

$$z = \frac{1}{2}\left[\ln 2 + i\left(\frac{\pi}{3} + 2n\pi\right)\right] = \frac{1}{2}\ln 2 + \frac{(6n+1)\pi i}{6}.$$

9 **3.** Calculate $f'(i/3)$ if $f(z) = \sqrt{3z - 1}$. Express answer(s) in exponential form.

Since $f'(z) = \dfrac{3}{2\sqrt{3z-1}}$,

$$f'(i/3) = \frac{3}{2\sqrt{3(i/3) - 1}} = \frac{3}{2\sqrt{-1 + i}} = \frac{3}{2\sqrt{\sqrt{2}e^{3\pi i/4}}} = \frac{3}{2^{5/4}e^{3\pi i/8}} = \frac{3}{2^{5/4}}e^{-3\pi i/8}.$$

10 **4.** (a) Show that the zeros of the function $f(z) = \tanh(3z+1)$ are $z = (-1 + n\pi i)/3$, where n is an integer.

(b) Calculate $f'(z)$ at the zeros in part (a) simplified as much as possible.

(c) What are the orders of the zeros in part (a)? Justify you answer.

(a) Since $\tanh(3z+1) = \dfrac{\sinh(3z+1)}{\cosh(3z+1)}$, zeros of $\tanh(3z+1)$ occur at the zeros of $\sinh(3z+1)$. Since $\sinh z$ has zeros $n\pi i$, where n is an integer, zeros of $\tanh(3z+1)$ are given by

$$3z + 1 = n\pi i \qquad \text{or} \qquad z = \frac{-1 + n\pi i}{3}.$$

(b) Since $f'(z) = 3\text{sech}^2(3z+1)$,

$$f'\left(\frac{-1+n\pi i}{3}\right) = 3\text{sech}^2(n\pi i) = \frac{3}{\cosh^2(n\pi i)} = \frac{3}{[\cos(n\pi)]^2} = \frac{3}{[(-1)^n]^2} = 3.$$

(c) Since $f(z)$ is analytic at its zeros, and $f((-1+n\pi i)/3) = 0$, but $f'((-1+n\pi i)/3) \neq 0$, the zeros are of order 1.

4 **5.** Evaluate the contour integral

$$\oint_C (z^3 + 4z + 1)\, dz,$$

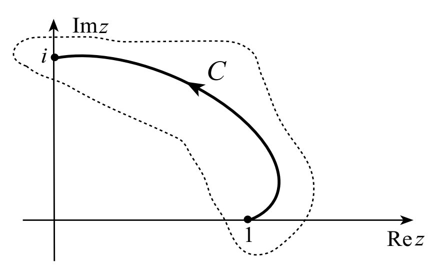
where C is the curve $|z| = 4$.

Since $z^3 + 4z + 1$ has indefinite integral $z^4/4 + 2z^2 + z + C$ in the whole complex plane, the contour interal is independent of path in the plane. Because the curve is closed, the contour integal has value 0.

8 **6.** Evaluate the contour integral

$$\int_C \frac{1}{z-4}\, dz,$$

where C is the curve in the figure to the right.



An antiderivative of $1/(z-4)$ in the domain shown is $\log_0(z-4)$. Hence

$$\begin{aligned}
\int_C \frac{1}{z-4}\, dz &= \{\log_0(z-4)\}_1^i = \log_0(i-4) - \log_0(-3) \\
&= [\ln|i-4| + i\arg(i-4)] - [\ln|-3| + i\arg(-3)] \\
&= \ln\sqrt{17} + i[\pi - \text{Tan}^{-1}(1/4)] - \ln 3 - i(\pi) \\
&= \left(\frac{1}{2}\ln 17 - \ln 3\right) - i\text{Tan}^{-1}(1/4).
\end{aligned}$$