Hyperbolic Functions

Certain combinations of the exponential function occur so often in physical applications that they are given special names. Specifically, half the difference of e^x and e^{-x} is defined as the **hyperbolic sine function** and half their sum is the **hyperbolic cosine function**. These functions are denoted as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}. \tag{1.1}$$

The names of the hyperbolic functions and their notations bear a striking resemblance to those for the trigonometric functions, and there are reasons for this. First, the hyperbolic functions $\sinh x$ and $\cosh x$ are related to the curve $x^2 - y^2 = 1$, called the *unit hyperbola*, in much the same way as the trigonometric functions $\sin x$ and $\cos x$ are related to the unit circle $x^2 + y^2 = 1$. Second, for each identity satisfied by the trigonometric functions, there is a corresponding identity satisfied by the hyperbolic functions — not the same identity, but one very similar. For example, using equations 1.1, we have

$$\begin{aligned}(\cosh x)^2 - (\sinh x)^2 &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{1}{4}\left[\left(e^{2x} + 2 + e^{-2x}\right) - \left(e^{2x} - 2 + e^{-2x}\right)\right] \\ &= 1.\end{aligned}$$

Thus the hyperbolic sine and cosine functions satisfy the identity

$$\cosh^2 x - \sinh^2 x = 1, \tag{1.2}$$

which is reminiscent of the identity $\cos^2 x + \sin^2 x = 1$ for the trigonometric functions.

Just as four other trigonometric functions are defined in terms of $\sin x$ and $\cos x$, four corresponding hyperbolic functions are defined as follows:

$$\begin{aligned}\tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad & \coth x &= \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \\ \operatorname{sech} x &= \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \qquad & \operatorname{csch} x &= \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.\end{aligned} \tag{1.3}$$

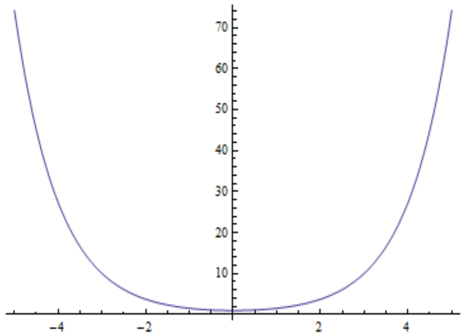
These definitions and 1.2 immediately imply that

$$1 - \tanh^2 x = \operatorname{sech}^2 x, \tag{1.4a}$$

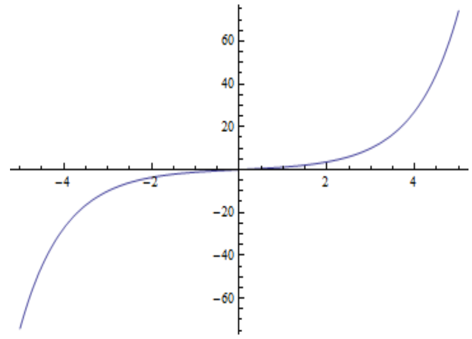
$$\coth^2 x - 1 = \operatorname{csch}^2 x, \tag{1.4b}$$

analogous to $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \csc^2 x$ respectively.

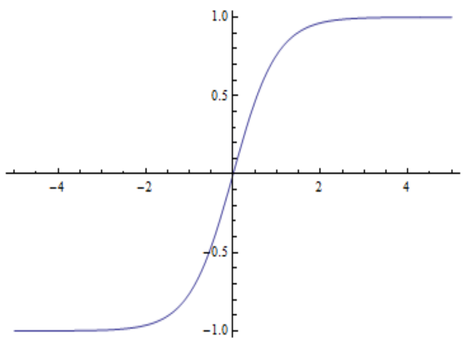
Graphs of the six hyperbolic functions are shown below. The functions $\cosh x$ and $\mathrm{sech} x$ are even, the other four are odd.



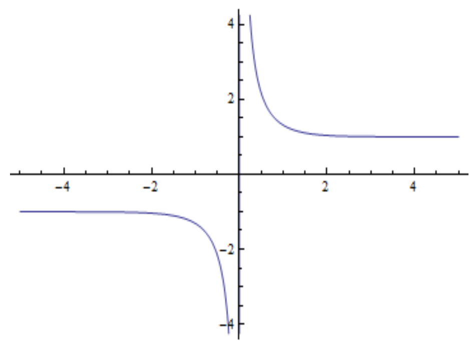
$\cosh x$



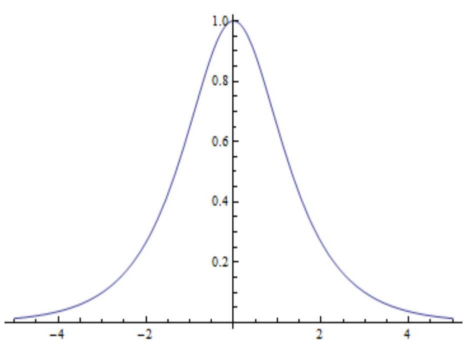
$\sinh x$



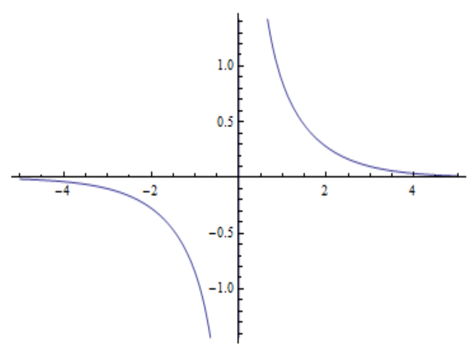
$\tanh x$



$\mathrm{coth} x$



$\mathrm{sech} x$



$\mathrm{csch} x$

Most trigonometric identities can be derived from the compound-angle formulas for $\sin(A \pm B)$ and $\cos(A \pm B)$. It is easy to verify similar formulas for the hyperbolic functions:

$$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B, \tag{1.5a}$$

$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B. \tag{1.5b}$$

For example, equations 1.1 give

$$
\begin{aligned}
\cosh A \cosh B - \sinh A \sinh B &= \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) \\
&\quad - \left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^B - e^{-B}}{2}\right) \\
&= \frac{1}{4}\big[\big(e^{A+B} + e^{A-B} + e^{B-A} + e^{-A-B}\big) \\
&\quad - \big(e^{A+B} - e^{A-B} - e^{B-A} + e^{-A-B}\big)\big] \\
&= \frac{1}{2}\left[e^{A-B} + e^{-(A-B)}\right] \\
&= \cosh{(A-B)}.
\end{aligned}
$$

With these formulas, we can derive hyperbolic identities analogous to many trigonometric identities:

$$
\begin{aligned}
\tanh{(A \pm B)} &= \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}, && (1.5c) \\
\sinh 2A &= 2 \sinh A \cosh A, && (1.5d) \\
\cosh 2A &= \cosh^2 A + \sinh^2 A && (1.5e) \\
&= 2\cosh^2 A - 1 && (1.5f) \\
&= 1 + 2\sinh^2 A, && (1.5g) \\
\tanh 2A &= \frac{2\tanh A}{1 + \tanh^2 A}, && (1.5h) \\
\sinh A \sinh B &= \frac{1}{2}\cosh{(A+B)} - \frac{1}{2}\cosh{(A-B)}, && (1.5i) \\
\sinh A \cosh B &= \frac{1}{2}\sinh{(A+B)} + \frac{1}{2}\sinh{(A-B)}, && (1.5j) \\
\cosh A \cosh B &= \frac{1}{2}\cosh{(A+B)} + \frac{1}{2}\cosh{(A-B)}, && (1.5k) \\
\sinh A + \sinh B &= 2\sinh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right), && (1.5l) \\
\sinh A - \sinh B &= 2\cosh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right), && (1.5m) \\
\cosh A + \cosh B &= 2\cosh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right), && (1.5n) \\
\cosh A - \cosh B &= 2\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right). && (1.5o)
\end{aligned}
$$