MATH 3810 Midterm Examination 1 February 1, 2012

Values

15 **1.** Find the solution to the partial differential equation

$$4u_y + u\,u_x = 4$$

that contains the curve $y + x^2 = 0$, $u = -2x$. Give the solution in explicit form.

C-equations are

$$\frac{dx}{u} = \frac{dy}{4} = \frac{du}{4}.$$

These gives the 2-parameter family of C-curves $u = y + \alpha$ and $x = u^2/8 + \beta$. A 1-parameter family of C-curves is

$$u = y + \alpha, \qquad x = \frac{u^2}{8} + \beta(\alpha).$$

For this family to pass through the initial curve for a specific value of α, we must have

$$-2x = -x^2 + \alpha, \quad x = \frac{4x^2}{8} + \beta.$$

These give

$$\alpha = x^2 - 2x, \quad \text{and} \quad \beta = x - \frac{x^2}{2} = \frac{1}{2}(2x - x^2) = -\frac{\alpha}{2}.$$

The 1-parameter family of C-curves generating the solution surface is

$$u = y + \alpha, \quad x = \frac{u^2}{8} - \frac{\alpha}{2}.$$

The solution surface is therefore defined implicitly by

$$x = \frac{u^2}{8} - \frac{1}{2}(u - y).$$

By writing this in the form $u^2 - 4u + (4y - 8x) = 0$, we can get the solution explicitly,

$$u = \frac{4 \pm \sqrt{16 - 4(4y - 8x)}}{2} = 2 \pm \sqrt{4 - 4y + 8x}.$$

To satisfy the initial curve, we must choose $u = 2 - 2\sqrt{1 - y + 2x}$.

10 **2.** Temperature $U(r,\theta,t)$ in a semi-circular plate with radius R, insulated top and bottom, must satisfy the partial differential equation

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right), \quad 0 < r < R, \quad -\pi/2 < \theta < \pi/2, \quad t > 0,$$

when there is no internal heat generation. The temperature in the plate at time $t = 0$ is only a function of $f(r)$ of the radial coordinate r. The semi-circular edge is held at temperature $g(\theta)$ for $t > 0$, and edge $\theta = -\pi/2$ is held at temperature $h(r)$. Edge $\theta = \pi/2$ is insulated. What are the boundary and initial conditions for $U(r,\theta,t)$?

The initial condition is

$$U(r,\theta,0) = f(r), \quad 0 < r < R, \quad -\pi/2 < \theta < \pi/2.$$

Two of the boundary conditions are

$$\begin{aligned} U(R,\theta,t) &= g(\theta), \quad -\pi/2 < \theta < \pi/2, \quad t > 0, \\ U(r,-\pi/2,t) &= h(r), \quad 0 < r < R, \quad t > 0. \end{aligned}$$

The boundary condition along the positive y-axis is $-\dfrac{\partial U}{\partial x} = 0$, but this must be expressed in terms of derivatives with respect to r and/or θ. To do this, we write that

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

The equations $r = \sqrt{x^2+y^2}$ and $\theta = \mathrm{Tan}^{-1}(y/x) \pm \pi$ give

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta, \\ \frac{\partial \theta}{\partial x} &= \frac{1}{1+y^2/x^2}(-y/x^2) = \frac{-y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}. \end{aligned}$$

Thus,

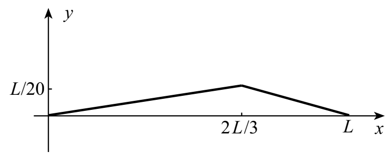
$$\frac{\partial U}{\partial x} = \cos\theta\frac{\partial U}{\partial r} - \frac{\sin\theta}{r}\frac{\partial U}{\partial \theta}.$$

Along $\theta = \pi/2$, the boundary condition is

$$0 = \frac{1}{r}\frac{\partial U}{\partial \theta} \quad \Longrightarrow \quad \frac{\partial U(r,\pi/2,t)}{\partial \theta} = 0, \quad 0 < r < R, \quad t > 0.$$

15 **3.** (a) A taut string of length L has its ends at $x = 0$ and $x = L$. The tension τ and linear density ρ of the string are constant. If gravity acting on the string is taken into account, what is the PDE for vertical displacements of the string? Identify any physical constants in your equation.

(b) If the string is released from rest at the position in the figure below, what are the initial conditions for $y(x,t)$?



(c) If the left end of the string is fixed on the x-axis, but the right end is free to move vertically, what are the boundary conditions for $y(x,t)$?

(d) Find and solve the problem for static deflections of the string. Draw its graph.

(a) The PDE is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - g, \quad 0 < x < L, \quad t > 0, \quad \text{where } c^2 = \tau/\rho \text{ and } g = 9.81.$$

(b) The initial conditions are

$$y(x,0) = \begin{cases} 3x/40, & 0 < x \le 2L/3, \\ 3(L-x)/20, & 2L/3 < x < L, \end{cases}$$
$$y_t(x,0) = 0, \quad 0 < x < L.$$

(c) The boundary conditions are

$$y(0,t) = 0, \quad t > 0, \qquad \text{and} \qquad y_x(L,t) = 0, \quad t > 0.$$

(d) Static deflections must satisfy

$$0 = c^2 \frac{d^2 y}{dx^2} - g, \quad 0 < x < L,$$
$$0 = y(0) = y'(L).$$

A general solution of the ODE is

$$y(x) = \frac{gx^2}{2c^2} + Ax + B.$$

The boundary conditions require

$$0 = B, \quad 0 = \frac{gL}{c^2} + A.$$

Thus,

$$y(x) = \frac{gx^2}{2c^2} - \frac{gLx}{c^2}.$$