MATH 3810 Midterm Examination 2 Solutions March 16, 2012

Values

12 **1.** The function $f(x)$, $0 \le x \le 2$ in the left figure below is to be represented in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$$

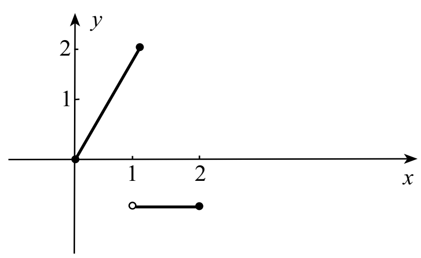
for some choice of L and constants a_n.

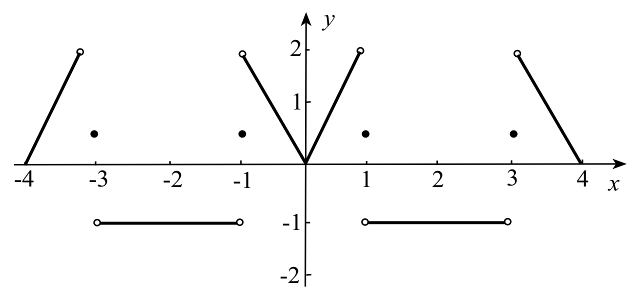
(a) What is the value of L?

(b) Find a_0.

(c) Set up, but do **NOT** evaluate, definite integrals for the values of the a_n, for $n > 0$.

(d) In the right figure for $-4 \le x \le 4$, draw the function to which the series in part (a) converges.





(a) $L = 2$

(b) $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 2x\,dx + \int_1^2 -dx = \left\{x^2\right\}_0^1 - \{x\}_1^2 = 0$

(c) $a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 2x\cos\frac{n\pi x}{2}\,dx + \int_1^2 -\cos\frac{n\pi x}{2}\,dx$

(d) The graph is shown above.

28 **2.** A cylindrical rod of length L and insulated sides is placed along the x-axis between $x = 0$ and $x = L$. At time $t = 0$, it is at temperature 0°C throughout. For time $t > 0$, its ends continue to be held at temperature 0°C. At every point in the rod, heat is generated at the constant rate of $g(x,t) = 1$. Find the temperature in the rod for $0 < x < L$ and $t > 0$.

The initial boundary value problem for temperature $U(x,t)$ is

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}, \quad 0 < x < L, \quad t > 0,\\
U(0,t) &= 0, \quad t > 0,\\
U(L,t) &= 0, \quad t > 0,\\
U(x,0) &= 0, \quad 0 < x < L.
\end{aligned}$$

Because the nonhomogeneity is time-independent, we set $U(x,t) = V(x,t) + \psi(x)$, where $\psi(x)$ is the steady-state solution, satisfying

$$\begin{aligned} 0 &= k\frac{d^2\psi}{dx^2} + \frac{k}{\kappa}, \quad 0 < x < L, \\ \psi(0) &= 0 = \psi(L). \end{aligned}$$

A general solution of the differential equation is $\psi(x) = -\dfrac{x^2}{2\kappa} + Ax + B$. The boundary conditions require

$$0 = \psi(0) = B, \qquad 0 = \psi(L) = -\frac{L^2}{2\kappa} + AL.$$

Thus, $\psi(x) = -\dfrac{x^2}{2\kappa} + \dfrac{Lx}{2\kappa} = \dfrac{x(L-x)}{2\kappa}$. The function $V(x,t)$ satisfies the homogeneous problem

$$\begin{aligned} \frac{\partial V}{\partial t} &= k\frac{\partial^2 V}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\ V(0,t) &= 0, \quad t > 0, \\ V(L,t) &= 0, \quad t > 0, \\ V(x,0) &= -\psi(x), \quad 0 < x < L. \end{aligned}$$

We search for separated functions $V(x,t) = X(x)T(t)$ satisfying the PDE and the boundary conditions. The PDE requires

$$XT' = kX''T \quad \Longrightarrow \quad \frac{X''}{X} = \frac{T'}{kT} = \alpha.$$

Thus, $X(x)$ and $T(t)$ satisfy the ODEs

$$X'' - \alpha X = 0 \qquad \text{and} \qquad T' - k\alpha T = 0.$$

The boundary conditions require $X(0) = 0$ and $X(L) = 0$. If we set $\alpha = -\lambda^2$, then $X(x)$ satisfies

$$\begin{aligned} X'' + \lambda^2 X &= 0, \quad 0 < x < L, \\ X(0) &= 0 = X(L). \end{aligned}$$

A general solution of the differential equation is $X(x) = A\cos\lambda x + B\sin\lambda x$. The boundary conditions require

$$0 = X(0) = A, \qquad 0 = X(L) = B\sin\lambda L.$$

Since $B \neq 0$, we set $\sin\lambda L = 0$, in which case $\lambda L = n\pi$, where n is an integer. Thus, values of λ are $\lambda_n = n\pi/L$, and $X(x) = B\sin(n\pi x/L)$. Correspondingly, $T(t) = Ce^{-n^2\pi^2kt/L^2}$. Separated functions are therefore

$$X(x)T(t) = be^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.$$

Because the PDE and boundary conditions are linear and homogeneous, we superpose separated functions and take

$$V(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.$$

The iniitial condition requires

$$-\psi(x) = -\frac{x(L-x)}{2\kappa} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad 0 < x < L.$$

The b_n are therefore coefficients in the Fourier sine series of the odd, $2L$-periodic extension of $-\psi(x)$; that is,

$$b_n = \frac{2}{L}\int_0^L -\frac{x(L-x)}{2\kappa}\sin\frac{n\pi x}{L}\,dx.$$

Integration by parts leads to

$$b_n = \frac{2L^2[(-1)^n - 1]}{n^3\pi^3\kappa}.$$

Thus,

$$\begin{aligned} V(x,t) &= \sum_{n=1}^{\infty}\frac{2L^2[(-1)^n-1]}{n^3\pi^3\kappa}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L} \\ &= -\frac{4L^2}{\pi^3\kappa}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-(2n-1)^2\pi^2kt/L^2}\sin\frac{(2n-1)\pi x}{L}. \end{aligned}$$

Finally, $U(x,t) = V(x,t) + \psi(x)$.