3.1

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + f''(x_i)\frac{h^2}{2} + O(h^3) \quad (1)$$

$$f(x_{i+2}) = f(x_i) + f'(x_i)(2h) + f''(x_i)\frac{(2h)^2}{2} + O(h^3) \quad (2)$$

(a) To eliminate $f'(x_i)$ terms subtract 2 times Eqn (1) from Eqn (2)

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)\left[\frac{4h^2}{2} - \frac{2h^2}{2}\right] + O(h^3)$$

$$\Rightarrow f''(x_i)\,h^2 = f(x_{i+2}) - 2f(x_{i+1}) + f(x_i) + O(h^3)$$

$$\Rightarrow \boxed{f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)}$$

(b) To eliminate $f''(x_i)$ terms subtract 4 times Eqn (1) from Eqn (2)

$$f(x_{i+2}) - 4f(x_{i+1}) = -3f(x_i) + f'(x_i)(2h - 4h) + O(h^3)$$

$$\Rightarrow f'(x_i)\,2h = -f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i) + O(h^3)$$

$$\rightarrow \boxed{f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)}$$

3.2

- Eliminate f_i''; f_i'', and f_i'''' terms
- Substracting the first two equations and the last two equations will eliminate f_i^{4} and f_i'''' in each and leave two equations with f_i' and f_i'''. Then f_i''' can be eliminated from those two equations.

- Eqn (1b) $-$ Eqn (2b)

$$f_{i+1} - f_{i-1} = 2hf_i' + \frac{h^3}{3} f_i''' + O(h^5) \qquad (5b)$$

- Eqn (3b) $-$ Eqn (4b)

$$f_{i+2} - f_{i-2} = 4hf_i' + \frac{8}{3} h^3 f_i''' + O(h^5) \qquad (6b)$$

- $8 *$ Eqn (5b) $-$ Eqn (6b)

$$8f_{i+1} - 8f_{i-1} - f_{i+2} + f_{i-2} = 12hf_i' + O(h^5)$$

Solve for f_i'

$$\boxed{f_i' = \frac{(f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2})}{12h} + O(h^4)}$$

3.3

Want an expression for f_i'' → must manipulate the given equations to eliminate all other derivatives

Step 1 Eq. (1) + Eq. (2) (Eliminate odd power derivatives)

$$\Rightarrow \quad f_{i+1} + f_{i-1} = 2f_i + h^2 f_i'' + \frac{h^4}{12} f_i'''' + O(h^6) \qquad (5)$$

Step 2 Eq. (3) + Eq. (4) (Eliminate odd power derivatives)

$$f_{i+2} + f_{i-2} = 2f_i + 4h^2 f_i'' + \frac{4}{3} h^4 f_i'''' + O(h^6) \qquad (6)$$

Step 3 Eliminate f_i'''' derivatives between Eq. (5) & Eq (6)

16 * Eq (5) − Eq. (6)

$$16 f_{i+1} + 16 f_{i-1} - f_{i+2} - f_{i-2} = 32 f_i + 16 h^2 f_i'' + \frac{16}{12} h^4 f_i'''' - 2 f_i - 4h^2 f_i'' - \frac{16}{12} h^4 f_i'''' + O(h^6)$$

$$\Rightarrow 16 f_{i+1} + 16 f_{i-1} - f_{i+2} - f_{i-2} = 30 f_i + 12 h^2 f_i'' + O(h^6)$$

$$\Rightarrow \quad f_i'' = \frac{-f_{i-2} + 16 f_{i-1} - 30 f_i + 16 f_{i+1} - f_{i+2} + O(h^6)}{12 h^2}$$

$$\boxed{f_i'' = \frac{-f_{i-2} + 16 f_{i-1} - 30 f_i + 16 f_{i+1} - f_{i+2}}{12 h^2} + O(h^4)}$$

3.4

$$f(x) = \frac{1}{x}$$

$$f'(x) = \frac{-1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

predict $f(5)$ using a base point at $x = 4$.

True Value: $f(5) = \frac{1}{5} = 0.2$

$x_i = 4$
$x_{i+1} = 5$
$h = 1$

<u>Zero order</u>

$$f(x_{i+1}) = f(x_i)$$

$$f(5) = \frac{1}{4} = 0.25 \leftarrow$$

$$\epsilon_t = \frac{0.2 - 0.25}{0.2} \times 100\% = -25\% \leftarrow$$

<u>first order</u>

$$f(x_{i+1}) = f(x_i) + f'(x_i)\,h$$

$$f(5) = \frac{1}{4} + \left(\frac{-1}{4^2}\right) 1 = 0.1875 \leftarrow$$

$$\epsilon_t = \frac{0.2 - 0.1875}{0.2} \times 100\% = +6.25\% \leftarrow$$

<u>second order</u>

$$f(x_{i+1}) = f(x_i) + f'(x_i)\,h + f''(x_i)\,\frac{h^2}{2}$$

$$f(5) = \frac{1}{4} + \left(\frac{-1}{4^2}\right) 1 + \left(\frac{2}{4^3}\right)\frac{(1)^2}{2}$$

$$f(5) = 0.203125 \leftarrow$$

$$\epsilon_t = \frac{0.2 - 0.203125}{0.2} \times 100\% = -1.56\% \leftarrow$$

3.5

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = \frac{-1}{x^2}$

2nd order Taylor Series expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x)$$

Approximate $f(5)$ from a base point of $x=3$:

- base point $x=3$
- spacing: $h=2$
- exact value $f(5) = \ln 5 = 1.60944$
- Approximation: $f(5) \simeq \ln 3 + 2\left(\frac{1}{3}\right) + \frac{(2)^2}{2}\left(\frac{-1}{(3)^2}\right)$

$$f(5) \simeq 1.09861 + 0.666667 - 0.222222$$

$$f(5) = 1.54306$$

$$\epsilon_t = \frac{(1.60944 - 1.54306)}{1.60944} \times 100 = \underline{4.12\%}$$

Use standard finite difference equations in one dimension for the partial derivative.

$$\theta^*(x^*,t^*) = [C_1 \exp(-\zeta_1^2 t^*)] \cos(\zeta_1 x^*)$$

$C_1 = 1.0472$ $\quad t^* = 5.0$
$\zeta_1 = 0.5$ (rad)

Backward Difference (First Order)

Jaluria pg 84 $\quad f_i' = \dfrac{(f_i - f_{i-1})}{\Delta x}$

in this nomenclature

$$\left.\frac{\partial\theta^*}{\partial x^*}\right|_{\substack{x^*=1 \\ \text{1st order, BD}}} = \frac{\theta^*(1,t^*) - \theta^*(1-\Delta x^*,t^*)}{\Delta x^*}$$

In all cases $[C_1 \exp(-\zeta_1^2 t^*)]$ is constant

$$[C_1 \exp(-\zeta_1^2 t^*)] = [1.0472 \exp(-0.5)^2 5.0] = 0.30003$$

Step size $\Delta x^* = 0.2$

$$\left.\frac{\partial\theta^*}{\partial x^*}\right|_{\substack{x^*=1 \\ \Delta x^*=0.2}} = \frac{\theta^*(1,5) - \theta^*(0.8,5)}{0.2}$$

$$\theta^*(1,5) = 0.3003 \cos(0.5(1)) = 0.263301$$

$$\theta^*(0.8,5) = 0.3003 \cos(0.5(0.8)) = 0.276346$$

$$\left.\frac{\partial\theta^*}{\partial x^*}\right|_{\substack{x^*=1 \\ \Delta x^*=0.2}} = \frac{0.263301 - 0.276346}{0.2} = -0.065225$$

CONTINUED

- step size $\Delta x^* = 0.1$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{\substack{x^*=1 \\ \Delta x^*=0.1}} = \frac{\theta^*(1,5) - \theta^*(0.9,5)}{0.1}$$

$$\theta^*(0.9,5) = 0.30003 \cos(0.5(0.9)) = 0.270161$$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{\substack{x^*=1 \\ \Delta x^*=0.1}} = \frac{0.263301 - 0.270161}{0.1} = -0.068600 \leftarrow$$

Exact Solution

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} = \left[c_1 \exp(-\zeta_1^2 t^*)\right] \left.\frac{\partial}{\partial x^*}(\cos(\zeta_1 x^*))\right|_{x^*=1}$$

$$= 0.30003\left[-\zeta_1 \sin(\zeta_1 x^*)\right]_{x^*=1}$$

$$= -0.30003(0.5)\sin(0.5) = \underline{-0.0719210}$$

Errors

$\Delta x^* = 0.2$: $$\epsilon_t = \frac{(-0.0719210 - (-0.065225))}{-0.0719210} \times 100\% = \underline{9.31\%}$$

$\Delta x^* = 0.1$: $$\epsilon_t = \frac{-0.0719210 - (-0.068600)}{-0.0719210} \times 100\% = \underline{4.62\%}$$

3.7

$$\theta(x) = \frac{\cosh[m(L-x)]}{\cosh[mL]}$$

$m = 150.0\,[m^{-1}]$
$L = 0.3\,[m]$

$\cosh[u] = \frac{1}{2}(e^{u} + e^{-u})$ or directly available on a Calculator

$$\frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{\cosh[m(L-x)]}{\cosh[mL]}\right) = \frac{-m \sinh[m(L-x)]}{\cosh[mL]}$$

First derivative forward difference approximation of $\frac{d\theta}{dx}$ is

$$\frac{d\theta}{dx} \simeq \frac{\theta_{x+\Delta x} - \theta_x}{\Delta x}$$

- $\Delta x = 0.05\,[m]$ $\qquad \frac{d\theta}{dx} \simeq \frac{\theta(0.05) - \theta(0.0)}{0.05}$

$$\theta(0.05) = \frac{\cosh[150(0.3-0.05)]}{\cosh[150(0.3)]} = \frac{\cosh[37.5]}{\cosh[45.0]}$$

$$= \frac{9.66080 \times 10^{15}}{1.746714 \times 10^{19}} = 0.00055308$$

$$\theta(0.0) = \frac{\cosh[mL]}{\cosh[mL]} = 1.0$$

$$\left.\frac{d\theta}{dx}\right|_{\Delta x = 0.05} = \frac{0.00055308 - 1.0}{0.05} = -19.9889$$

$$\left.\frac{d\theta}{dx}\right|_{exact} = \frac{1}{\cosh[mL]}\left\{\sinh[m(L-x)] \; -m\right\} = \frac{-m}{\cosh[mL]}\sinh[m(L-x)]$$

$$\left.\frac{d\theta}{dx}\right|_{exact,\,x=0} = -\frac{m \sinh(mL)}{\cosh(mL)} = -150.0 \tanh(45.0) = -150.0\,(1.0) = \underline{-150.}$$

$$\varepsilon_t = \frac{-150.0 - (-19.9889)}{-150.0} \times 100\% = \underline{86.67\%}$$

CONTINUED

3.7 (continued)

$$\left.\frac{d\theta}{dx}\right|_{\Delta x=0.01} = \frac{\theta(0.01) - \theta(0.0)}{0.01}$$

$$\theta(0.01) = \frac{\cosh[150(0.3-0.01)]}{\cosh[45.0]} = \frac{\cosh(43.5)}{\cosh(45.0)}$$

$$= \frac{3.89744 \times 10^{18}}{1.746714 \times 10^{19}} = 0.22313$$

$$\left.\frac{d\theta}{dx}\right|_{\Delta x=0.01} = \frac{0.22313 - 1.0}{0.01} = -77.687$$

$$\epsilon_t = \frac{-150.0 - (-77.687)}{-150.0} \times 100\% = \underline{48.21}\ \%$$

3.8

$$a \frac{d^2T}{dx^2} + b(c - T) = 0$$

Derive discretization equation by integrating over the typical "T_P" control volume shown in the diagram.

$$\int_{\forall} a \frac{d^2T}{dx^2} \, d\forall + \int_{\forall} b(c-T) \, d\forall = 0$$

$$a \int_A \int_w^e \frac{d}{dx}\left(\frac{dT}{dx}\right) dx \, dA + \int_A \int_w^e (bc - bT) \, dx \, dA = 0$$

For constant cross-sectional area A and values assumed constant over the C.V. faces and simplifying

$$aA\left[\left(\frac{dT}{dx}\right)_e - \left(\frac{dT}{dx}\right)_w\right] + bcA\int_w^e dx - bA\int_w^e T\,dx = 0$$

Assume a piecewise linear profile for T for $\frac{dT}{dx}$ terms

$$\Rightarrow \left(\frac{dT}{dx}\right)_e \simeq \frac{(T_E - T_P)}{(\delta x)_e} \quad \text{and} \quad \left(\frac{dT}{dx}\right)_w \simeq \frac{(T_P - T_W)}{(\delta x)_w}$$

CONTINUED

3.8 (continued)

Assume a stepwise profile for the $\int T dx$ term

$$\int_w^e T dx \approx T_P \int_w^e dx$$

$$\Rightarrow aA\left[\frac{(T_E - T_P)}{(\delta x)_e} - \frac{(T_P - T_W)}{(\delta x)_w}\right] + bcA(\Delta x) - bAT_P(\Delta x) = 0$$

$$\Rightarrow \frac{aA}{(\delta x)_e} T_E + \frac{aA}{(\delta x)_w} T_W - \left(\frac{aA}{(\delta x)_e} + \frac{aA}{(\delta x)_w}\right) T_P + bcA(\Delta x) - bA(\Delta x)T_P = 0$$

$$\Rightarrow A\left(\frac{a}{(\delta x)_e} + \frac{a}{(\delta x)_e} + b(\Delta x)\right) T_P = \frac{Aa}{(\delta x)_e} T_E + \frac{Aa}{(\delta x)_w} T_W + bcA(\Delta x)$$

dividing through by A yields

$$\left[\frac{a}{(\delta x)_e} + \frac{a}{(\delta x)_w} + b(\Delta x)\right] T_P = \frac{a}{(\delta x)_e} T_E + \frac{a}{(\delta x)_w} T_W + bc(\Delta x)$$

$$\Rightarrow \boxed{\begin{aligned} a_P &= a_E + a_W + b(\Delta x) \\ a_E &= \frac{a}{(\delta x)_e} \\ a_W &= \frac{a}{(\delta x)_w} \\ b_P &= bc(\Delta x) \end{aligned}}$$

(or with A multiplying all terms)

3.9

(a) $$\underbrace{\int_{\forall} \frac{d^2\Phi}{dx^2}\, d\forall}_{I} - \underbrace{\int_{\forall} D x^2 \Phi\, d\forall}_{II} = 0$$

Assume piecewise linear profile for term I

$$I = A_e \frac{d\Phi}{dx}\bigg|_e - A_w \frac{d\Phi}{dx}\bigg|_w = A_e \frac{(\Phi_E - \Phi_P)}{\Delta x} - A_w \frac{(\Phi_P - \Phi_W)}{\Delta x}$$

$$= \frac{A}{\Delta x}\Phi_E + \frac{A}{\Delta x}\Phi_W - \frac{2A}{\Delta x}\Phi_P$$

$$II = -D \int_A \int_w^e x^2 \Phi \, dx\, dA$$

Assume stepwise profile for Φ in term II

$$\Rightarrow II = -D \int_A \Phi_P \int_w^e x^2 dx\, dA = -D \int_A \Phi_P \frac{x^3}{3}\bigg|_w^e dA$$

$$II = -D \int_A \Phi_P \frac{(x_e^3 - x_w^3)}{3} dA = -\frac{DA_P}{3}\Phi_P (x_e^3 - x_w^3)$$

$$II = -\frac{DA_P}{3}(x_e^3 - x_w^3)\Phi_P$$

Gather terms

$$\frac{A}{\Delta x}\Phi_E + \frac{A}{\Delta x}\Phi_W - \frac{2A}{\Delta x}\Phi_P - \frac{DA_P}{3}(x_e^3 - x_w^3)\Phi_P = 0$$

$$\left[\frac{2A}{\Delta x} + \frac{D}{3}A_P(x_e^3 - x_w^3)\right]\Phi_P = \frac{A}{\Delta x}\Phi_E + \frac{A}{\Delta x}\Phi_W$$

$a_P = (2A/\Delta x) + DA_P(x_e^3 - x_w^3)/3$

$a_E = A/\Delta x$

$a_W = A/\Delta x$ $\quad\quad b_P = 0$