(a) $T_e = (\frac{1}{2} + \alpha_e) T_2 + (\frac{1}{2} - \alpha_e) T_3$

$$\alpha_e = \frac{(Pe)_e^2}{10 + 2(Pe)_e^2}$$

$$(Pe)_e = \frac{\dot{m}_e}{D_e} = \frac{(\rho U A)_e}{\left(\frac{k}{c_p}\frac{A}{\Delta x}\right)_e} = \frac{0.5136(0.01)(1)}{\left(\frac{0.0370}{2287}\right)\frac{1}{(0.02)}} = \frac{0.005136}{0.00080892}$$

$(Pe)_e = 6.3492$

Because of constant grid spacing, properties & velocity

$(Pe)_w = (Pe)_e = Pe = 6.3492$

$$\alpha_e = \frac{(6.3492)^2}{10 + 2(6.3492)^2} = 0.44483$$

$T_e = (\frac{1}{2} + 0.44483)(406.8096) + (\frac{1}{2} - 0.44483) 418.2758$

$T_e = 407.4422$ [K]

$\alpha_w = \alpha_e$

$T_w = (\frac{1}{2} + \alpha_w) T_1 + (\frac{1}{2} - \alpha_w) T_2$

$= (\frac{1}{2} + 0.44483) 405.0334 + (\frac{1}{2} - 0.44483) 406.8096$

$T_w = 405.1314$ [K]

CONTINUED

(b) $\left.\frac{dT}{dx}\right|_e = \beta_e \frac{(T_E - T_P)}{(\delta x)_e} = \beta_e \frac{(T_3 - T_2)}{\Delta x}$

$$\beta_e = \beta_w = \frac{1 + 0.005(Pe)^2}{1 + 0.05(Pe)^2}$$

$$\beta_e = \frac{1 + 0.005(6.3492)^2}{1 + 0.05(6.3492)^2} = 0.39845$$

$$\left.\frac{dT}{dx}\right|_e = 0.39845 \frac{(418.2758 - 406.8096)}{0.02}$$

$$\left.\frac{dT}{dx}\right|_e = 228.435 \left[\frac{K}{m}\right]$$

$$\left.\frac{dT}{dx}\right|_w = \beta_w \frac{(T_P - T_W)}{(\delta x)_w} = \beta_w \frac{(T_2 - T_1)}{\Delta x}$$

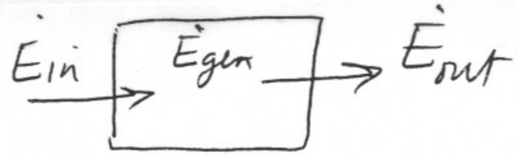
$$\left.\frac{dT}{dx}\right|_w = 0.39845 \frac{(406.8096 - 405.0334)}{0.02}$$

$$\left.\frac{dT}{dx}\right|_w = 35.3863 \left[\frac{K}{m}\right]$$

CONTINUED

7.1 (CONTINUED)

(C)



$$\dot{E}_{in} + \dot{E}_{gen} = \dot{E}_{out}$$

$$\dot{E}_{in} + \dot{E}_{gen} - \dot{E}_{out} = 0$$

$$\dot{E}_{in} = \dot{m}_w c_p T_w - k_w A_w \left.\frac{dT}{dx}\right|_w$$

$$= (0.005136)(2287)\,405.1314 - 0.0370\,(1)\,35.3863$$

$$\left[\frac{kg}{s}\right] \left[\frac{J}{kg\,K}\right] [K] \qquad \left[\frac{W}{m\,K}\right]\cdot m^2 \left[\frac{K}{m}\right]$$

$$[W] \checkmark \qquad [W] \checkmark$$

$$\dot{E}_{in} = 4758.686 - 1.30929 = 4757.377\,[W]$$

$$\dot{E}_{gen} = \dot{Q}'''(Vol)_p = 1000\left[\frac{W}{m^3}\right]\cdot 1\,[m^2]\cdot 0.02\,[m] = 20\,[W]$$

$$\dot{E}_{out} = \dot{m}_e C_p T_e - k_e A_e \left.\frac{dT}{dx}\right|_e$$

$$= 0.005136\,(2287)\,407.4422 - 0.0370\,(1)\,228.435$$

$$\dot{E}_{out} = 4785.8291 - 8.4521 = 4777.377\ [W]$$

Energy balance?

$$4757.377 + 20.0 - 4777.377 = 0 \checkmark$$

Balances ✓

7.2

(a) $T_e = \left(\frac{1}{2} + \alpha_e\right) T_P + \left(\frac{1}{2} - \alpha_e\right) T_E$

$$\alpha_e = \frac{1}{2} - \frac{\left(e^{\frac{1}{2}Pe} - 1\right)}{\left(e^{Pe} - 1\right)}$$

$$Pe = \frac{\dot{m}}{D} = \frac{\rho U A}{\frac{kA}{C_p \Delta x}} = \frac{\rho U \Delta x C_p}{k} = \frac{0.24883(-0.09)(0.01)(1207)}{0.0901}$$

$$\boxed{Pe = -3.00}$$

$$\alpha_e = \frac{1}{2} - \frac{\left(e^{-1.5} - 1\right)}{\left(e^{-3} - 1\right)} = -0.31757$$

$$\alpha_w = \alpha_e = -0.31757$$

$$T_e = \left(\frac{1}{2} + (-0.31757)\right) 703.8942 + \left(\frac{1}{2} - (-0.31757)\right) 702.1379$$

$$T_e = 702.4583 \; [K] \qquad \leftarrow T_e$$

$$T_w = \left(\frac{1}{2} + \alpha_w\right) T_W + \left(\frac{1}{2} - \alpha_w\right) T_P$$

$$T_w = \left(\frac{1}{2} + (-0.31757)\right) 725.0515 + \left(\frac{1}{2} - (-0.31757)\right) 703.8942$$

$$T_w = 707.7539 \; [K] \qquad \leftarrow T_w$$

CONTINUED

(b) $$\left.\frac{dT}{dx}\right|_e = \beta_e \frac{(T_E - T_P)}{(\delta x)_e} = \beta_e \frac{(T_3 - T_2)}{\Delta x}$$

$$\beta_e = \frac{Pe\, e^{\frac{1}{2}Pe}}{(e^{Pe} - 1)} = \frac{(-3.0)(e^{-1.5})}{(e^{-3.0} - 1)} = 0.70446$$

$$\left.\frac{dT}{dx}\right|_e = \frac{0.70446\,(702.1379 - 703.8942)}{0.01} = \underline{\underline{-123.724 \left[\frac{K}{m}\right]}}$$

$$\left.\frac{dT}{dx}\right|_w = \beta_w \frac{(T_P - T_W)}{(\delta x)_w} = \beta_w \frac{(T_2 - T_1)}{\Delta x}$$

$$\beta_w = \beta_e$$

$$\left.\frac{dT}{dx}\right|_w = \frac{0.70446\,(703.8942 - 725.0515)}{0.01} = \underline{\underline{-1490.45 \left[\frac{K}{m}\right]}}$$

(c) Energy balance

$$\dot{E}_{in} + \dot{E}_{gen} = \dot{E}_{out}$$

$$\dot{m} C_p T_w - k_w A_w \left.\frac{dT}{dx}\right|_w + \dot{Q}'''(Vol) \stackrel{?}{=} \dot{m} C_p T_e - k_e A_e \left.\frac{dT}{dx}\right|_e$$

CONTINUED

7.2 (CONTINUED)

$\dot{m} C_p T_w = 0.24883(-0.09)(1)(1207)[707.7539] = -19,130.87 \; [W]$

$-k_w A_w \frac{dT}{dx}\Big|_w = -(0.0901)(1)(-1490.45) = +134.29 \; [W]$

$\dot{Q}'''(Vol) = \dot{Q}''' A \Delta x = (2000)(1)(0.01) = +20 \; [W]$

$\dot{m} C_p T_e = 0.24883(-0.09)(1)(1207)[702.4583] = -18,987.73 \; [W]$

$-k_e A_e \frac{dT}{dx}\Big|_e = -(0.0901)(1)(-123.724) = +11.148 \; [W]$

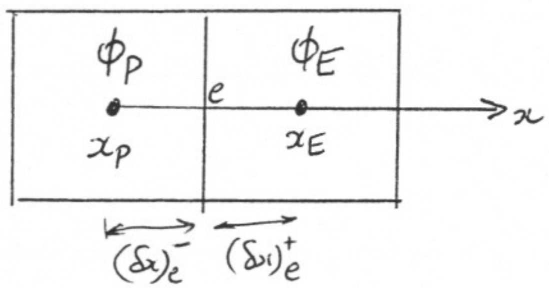
$-19,130.87 + 134.29 + 20 \stackrel{?}{=} -18,987.73 + 11.148$

$-18,976.58 \stackrel{?}{=} -18,976.58$

✓

Yes energy is Conserved..

(a) Find the expression for α_e by applying the exact solution between x_P & x_E



→ $\phi_e = \phi(x = x_e)$

$$\frac{(\phi_e - \phi_P)}{(\phi_E - \phi_P)} = \frac{\left[\exp\left\{(Pe)_e \frac{(x_e - x_P)}{(x_E - x_P)}\right\} - 1\right]}{\left[\exp\{(Pe)_e\} - 1\right]}$$

For uniform grid spacing $\quad \frac{x_e - x_P}{(x_E - x_P)} = \frac{(\delta x)_e^-}{(\delta x)_e^- + (\delta x)_e^+} = \frac{1}{2}$

$$\frac{(\phi_e - \phi_P)}{(\phi_E - \phi_P)} = \frac{\left[\exp\{\frac{1}{2}(Pe)_e\} - 1\right]}{\left[\exp\{(Pe)_e\} - 1\right]}$$

rearrange this expression and compare with the ϕ_e interpolation expression involving α_e

$$\phi_e = \phi_P + (\phi_E - \phi_P)\frac{\left[\exp\{\frac{1}{2}(Pe)_e\} - 1\right]}{\left[\exp\{(Pe)_e\} - 1\right]}$$

$$\phi_e = \left(1 - \frac{\left[\exp\{\frac{1}{2}(Pe)_e\} - 1\right]}{\left[\exp\{(Pe)_e\} - 1\right]}\right)\phi_P + \left(\frac{\left[\exp\{\frac{1}{2}(Pe)_e\} - 1\right]}{\left[\exp\{(Pe)_e\} - 1\right]}\right)\phi_E$$

CONTINUED

Compare the previous expression with

$$\phi_e = \left(\tfrac{1}{2}+\alpha_e\right)\phi_P + \left(\tfrac{1}{2}-\alpha_e\right)\phi_E$$

Page $7-\frac{8}{12}$

yields either

$$\left(\tfrac{1}{2}+\alpha_e\right) = 1 - \frac{\left[\exp\{\tfrac{1}{2}(Pe)_e\}-1\right]}{\left[\exp\{(Pe)_e\}-1\right]}$$

or

$$\left(\tfrac{1}{2}-\alpha_e\right) = \frac{\left[\exp\{\tfrac{1}{2}(Pe)_e\}-1\right]}{\left[\exp\{(Pe)_e\}-1\right]}$$

to get

$$\boxed{\alpha_e = \frac{1}{2} - \frac{\left[\exp\{\tfrac{1}{2}(Pe)_e\}-1\right]}{\left[\exp\{(Pe)_e\}-1\right]}}$$

(b) Check the limits (For simplicity drop the e subscript on Pe)

(i) $Pe \to 0$

$$\alpha_e \to \frac{1}{2} - \frac{(e^0-1)}{(e^0-1)} = \frac{1}{2} - \frac{(1-1)}{(1-1)} \; ?$$

Use L'Hôpital's Rule

$$\lim_{x\to 0}\frac{(e^{\frac{1}{2}x}-1)}{(e^x-1)} = \lim_{x\to 0}\frac{\frac{1}{2}e^{\frac{1}{2}x}}{e^x} = \frac{\frac{1}{2}e^0}{e^0} = \frac{1}{2}$$

$$\rightarrow \alpha_e \to \frac{1}{2} - \frac{1}{2} = 0 \quad \text{for } Pe \to 0$$

$$\boxed{\lim_{Pe\to 0}\alpha_e = 0}$$

CONTINUED

(ii) $Pe \to +\infty$

$$\alpha_e \to \frac{1}{2} - \frac{e^{+\frac{1}{2}\infty} - 1}{(e^{+\infty} - 1)} \to \frac{1}{2} - \frac{1}{\infty} \to \frac{1}{2} - 0 \Rightarrow \frac{1}{2}$$

$\alpha_e = \frac{1}{2}$ for $Pe \to +\infty$ $\qquad e^{+\infty} >> e^{+\frac{1}{2}\infty} >> 1$

$$\boxed{\lim_{Pe \to +\infty} \alpha_e = \frac{1}{2}}$$

(iii) $Pe \to -\infty$

$$\alpha_e \to \frac{1}{2} - \frac{[e^{-\frac{1}{2}\infty} - 1]}{[e^{-\infty} - 1]} \to \frac{1}{2} - \frac{[0-1]}{[0-1]} \to \frac{1}{2} - 1 \Rightarrow -\frac{1}{2}$$

$$\boxed{\lim_{Pe \to -\infty} \alpha_e = -\frac{1}{2}}$$

(c) Using the original interpolation for ϕ_e with the limiting values of α_e from part (b):

(i) $\alpha_e = 0$

$$\to \phi_e = \left(\tfrac{1}{2} + 0\right)\phi_P + \left(\tfrac{1}{2} - 0\right)\phi_E = \tfrac{1}{2}(\phi_P + \phi_E)$$

(average)

(ii) $\alpha_e = \frac{1}{2}$

$$\to \phi_e = \left(\tfrac{1}{2} + \tfrac{1}{2}\right)\phi_P + \left(\tfrac{1}{2} - \tfrac{1}{2}\right)\phi_E = \phi_P$$

(upwind value) $\phi_P \longrightarrow \phi_e$

(iii) $\alpha_e = -\frac{1}{2}$

$$\to \phi_e = \left(\tfrac{1}{2} + \left(-\tfrac{1}{2}\right)\right)\phi_P + \left(\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)\right)\phi_E = \phi_E$$

(upwind value) $\phi_e \longleftarrow \phi_E$

(a) $$\alpha = \frac{1}{2} - \frac{\left(e^{\frac{1}{2}Pe} - 1\right)}{\left(e^{Pe} - 1\right)}$$

(i) $Pe \to 0$ $$\lim_{Pe \to 0} \alpha = \frac{1}{2} - \lim_{Pe \to 0} \frac{\left(e^{\frac{1}{2}Pe} - 1\right)}{\left(e^{Pe} - 1\right)}$$

Need to Use L'Hôpital's Rule

$$\lim_{Pe \to 0} \frac{\left(e^{\frac{1}{2}Pe} - 1\right)}{\left(e^{Pe} - 1\right)} = \lim_{Pe \to 0} \frac{\frac{1}{2}e^{\frac{1}{2}Pe}}{e^{Pe}} = \frac{\frac{1}{2}(1)}{(1)} = \frac{1}{2}$$

$$\Rightarrow \lim_{Pe \to 0} \alpha = \frac{1}{2} - \frac{1}{2} = \boxed{0} \quad \leftarrow (i)$$

(ii) $Pe \to +\infty$ $$\lim_{Pe \to \infty} \alpha = \frac{1}{2} - \lim_{Pe \to \infty} \frac{\left(e^{\frac{1}{2}Pe} - 1\right)}{\left(e^{Pe} - 1\right)}$$

$$e^{\frac{1}{2}Pe} >> 1 \quad \text{for } Pe \to \infty$$

$$e^{Pe} >> 1 \quad \text{for } Pe \to \infty$$

$$\Rightarrow \lim_{Pe \to \infty} \frac{\left(e^{\frac{1}{2}Pe} - 1\right)}{\left(e^{Pe} - 1\right)} = \lim_{Pe \to \infty} \frac{1}{e^{\frac{1}{2}Pe}} \to \frac{1}{\infty} \to 0$$

$$\Rightarrow \lim_{Pe \to \infty} \alpha = \frac{1}{2} - 0 = +\frac{1}{2} \quad \leftarrow (ii)$$

CONTINUED

(iii) $\underline{Pe \to -\infty}$ $\lim_{Pe \to -\infty} \alpha = \frac{1}{2} - \lim_{Pe \to -\infty} \frac{(e^{\frac{1}{2}Pe} - 1)}{(e^{Pe} - 1)}$ 5/

$$\lim_{Pe \to -\infty} (e^{\frac{1}{2}Pe}) = 0$$

$$\lim_{Pe \to -\infty} e^{Pe} = 0$$

$$\Rightarrow \lim_{Pe \to -\infty} \alpha = \frac{1}{2} - \frac{(0-1)}{(0-1)} = \frac{1}{2} - 1 = -\frac{1}{2}$$ ← (iii)

b) $\underline{Pe \to 0}$ $\lim_{Pe \to 0} \beta = \lim_{Pe \to 0} \frac{Pe\, e^{\frac{1}{2}Pe}}{(e^{Pe} - 1)}$ ← need to use L'Hôpital's Rule

$$\lim_{Pe \to 0} \frac{Pe\, e^{\frac{1}{2}Pe}}{(e^{Pe} - 1)} = \lim_{Pe \to 0} \frac{(\frac{1}{2}Pe + 1)\, e^{\frac{1}{2}Pe}}{e^{Pe}} = \lim_{Pe \to 0} \frac{(\frac{1}{2}Pe + 1)}{e^{\frac{1}{2}Pe}}$$

$$= \frac{(\frac{1}{2}(0) + 1)}{(1)} = 1$$ ← (i)

CONTINUED

7.4 (CONTINUED)

(ii) $Pe \to +\infty$

$$\lim_{Pe\to\infty} \beta = \lim_{Pe\to\infty} \frac{Pe\, e^{\frac{1}{2}Pe}}{(e^{Pe}-1)}$$

$$e^{Pe} \gg 1$$

$$\Rightarrow \lim_{Pe\to\infty} \beta = \lim_{Pe\to\infty} \frac{Pe}{e^{\frac{1}{2}Pe}} \to \frac{1}{\infty} \to \underline{0} \leftarrow$$

because $e^{\frac{1}{2}Pe} \gg Pe$ for $Pe \to \infty$

(iii) $Pe \to -\infty$ $\quad \lim_{Pe\to-\infty} \beta = \lim_{Pe\to-\infty} \frac{Pe\, e^{\frac{1}{2}Pe}}{(e^{Pe}-1)}$

$$e^{Pe} \ll 1 \quad \text{for } Pe \to -\infty$$

$$\Rightarrow \lim_{Pe\to-\infty} \beta = \lim_{Pe\to-\infty} \left(-\frac{Pe}{e^{-\frac{1}{2}Pe}}\right) \to \frac{1}{\infty} \to \underline{0} \leftarrow \text{(iii)}$$

because $e^{\frac{1}{2}(-Pe)} \gg (-Pe)$ for $Pe \to -\infty$