Problems

3.1 Use the following Taylor series expansions:

$$f(x_{i+1}) = f(x_i) + f'(x_i)\, h + f''(x_i)\, \frac{h^2}{2} + \mathcal{O}(h^3) \tag{P3-1}$$

$$f(x_{i+2}) = f(x_i) + f'(x_i)\, 2h + f''(x_i)\, \frac{(2h)^2}{2} + \mathcal{O}(h^3) \tag{P3-2}$$

to answer the following questions.

(a) Derive the second forward difference equation for $f''(x_i)$. This is a first order approximation involving $f(x_i)$, $f(x_{i+1})$, and $f(x_{i+2})$).

(b) Derive a forward finite difference equation for $f'(x_i)$ that is a second order approximation (i.e. has truncation error of $\mathcal{O}(h^2)$).

3.2 Given the following Taylor series expansions:

$$f(x_{i+1}) = f(x_i) + h\ f'(x_i) + \frac{h^2}{2!}\ f''(x_i) + \frac{h^3}{3!}\ f'''(x_i) + \frac{h^4}{4!}\ f''''(x_i) + \mathcal{O}(h^5) \tag{P3-3}$$

$$f(x_{i-1}) = f(x_i) - h\ f'(x_i) + \frac{h^2}{2!}\ f''(x_i) - \frac{h^3}{3!}\ f'''(x_i) + \frac{h^4}{4!}\ f''''(x_i) + \mathcal{O}(h^5) \tag{P3-4}$$

$$f(x_{i+2}) = f(x_i) + 2h\ f'(x_i) + \frac{(2h)^2}{2!}\ f''(x_i) + \frac{(2h)^3}{3!}\ f'''(x_i) + \frac{(2h)^4}{4!}\ f''''(x_i) + \mathcal{O}(h^5) \tag{P3-5}$$

$$f(x_{i-2}) = f(x_i) - 2h\ f'(x_i) + \frac{(2h)^2}{2!}\ f''(x_i) - \frac{(2h)^3}{3!}\ f'''(x_i) + \frac{(2h)^4}{4!}\ f''''(x_i) + \mathcal{O}(h^5) \tag{P3-6}$$

The set of equations above can be written using a more simplified form (where $f_i = f(x_i)$ has been used) as:

$$f_{i+1} = f_i + h\ f_i' + \frac{h^2}{2}\ f_i'' + \frac{h^3}{6}\ f_i''' + \frac{h^4}{24}\ f_i'''' + \mathcal{O}(h^5) \tag{P3-7}$$

$$f_{i-1} = f_i - h\ f_i' + \frac{h^2}{2}\ f_i'' - \frac{h^3}{6}\ f_i''' + \frac{h^4}{24}\ f_i'''' + \mathcal{O}(h^5) \tag{P3-8}$$

$$f_{i+2} = f_i + 2h\ f_i' + 2h^2\ f_i'' + \frac{4}{3}h^3\ f_i''' + \frac{2}{3}h^4\ f_i'''' + \mathcal{O}(h^5) \tag{P3-9}$$

$$f_{i-2} = f_i - 2h\ f_i' + 2h^2\ f_i'' - \frac{4}{3}h^3\ f_i''' + \frac{2}{3}h^4\ f_i'''' + \mathcal{O}(h^5) \tag{P3-10}$$

Use the above equations (in either notation) to derive a centred finite difference (central difference) equation for $f'(x_i)$ (which is f_i' in the simpler notation) that is a fourth order approximation (i.e. has truncation error of $\mathcal{O}(h^4)$).

3.3 Given the following Taylor series expansions:

$$f_{i+1} = f_i + h\ f_i' + \frac{h^2}{2}\ f_i'' + \frac{h^3}{6}\ f_i''' + \frac{h^4}{24}\ f_i'''' + \frac{h^5}{120}\ f_i''''' + \mathcal{O}(h^6) \tag{P3-11}$$

$$f_{i-1} = f_i - h\ f_i' + \frac{h^2}{2}\ f_i'' - \frac{h^3}{6}\ f_i''' + \frac{h^4}{24}\ f_i'''' - \frac{h^5}{120}\ f_i''''' + \mathcal{O}(h^6) \tag{P3-12}$$

$$f_{i+2} = f_i + 2h\ f_i' + 2h^2\ f_i'' + \frac{4}{3}h^3\ f_i''' + \frac{2}{3}h^4\ f_i'''' + \frac{4}{15}h^5\ f_i''''' + \mathcal{O}(h^6) \tag{P3-13}$$

$$f_{i-2} = f_i - 2h\ f_i' + 2h^2\ f_i'' - \frac{4}{3}h^3\ f_i''' + \frac{2}{3}h^4\ f_i'''' - \frac{4}{15}h^5\ f_i''''' + \mathcal{O}(h^6) \tag{P3-14}$$

where $f_i = f(x_i)$ has been used to simplify the notation. Note also in this simplified notation that $f_{i+1} = f(x_{i+1})$, $f_{i+2} = f(x_{i+2})$, $f_{i-1} = f(x_{i-1})$, and $f_{i-2} = f(x_{i-2})$.

Use Equations (P3-11) to (P3-14) above to derive a centered finite difference (central difference) equation for f_i'' (i.e. $f''(x_i)$) that is a fourth order approximation (i.e. has truncation error of $\mathcal{O}(h^4)$). Show your work.

3.4 Use zero through second-order Taylor series expansions to predict $f(5)$ for $f(x) = 1/x$ using a base point of $x = 4$. Calculate the true percent relative error for each approximation.

3.5 Use a second-order Taylor series expansion to predict $f(5)$ for $f(x) = \ln x$ using a base point of $x = 3$. Calculate the true percent relative error for the approximation.

3.6 The approximate (one-term) analytical solution for time-varying temperature (in dimensionless form) for a plane wall is

$$\theta^*(x^*, t^*)\ =\ \left[C_1 \exp(-\zeta_1^2\, t^*)\right]\ \cos(\zeta_1\, x^*) \tag{P3-15}$$

Calculate $\dfrac{\partial \theta^*}{\partial x^*}$ at $x^* = 1$ using a first order backward difference for steps of $\Delta x^* = 0.2$ and $\Delta x^* = 0.1$ with $C_1 = 1.0472$, $\zeta_1 = 0.5$ [rad], and $t^* = 5$. Determine the true relative error in both cases.

3.7 In this problem, use the following expression for dimensionless temperature, $\theta(x)$:

$$\theta(x) = \frac{\cosh\left[m\left(L-x\right)\right]}{\cosh\left[mL\right]} \tag{P3-16}$$

where $m = 15.0\ [m^{-1}]$ and $L = 0.3\ [m]$. Calculate $\dfrac{d\theta}{dx}$ at $x = 0$ using a first order forward difference approximation for steps of $\Delta x = 0.05\ [m]$ and $\Delta x = 0.01\ [m]$. Determine the true relative error in both cases. Note that the hyperbolic cosine ($\cosh$) and its derivative are in Appendix B.1 of Incropera *et al.* [1].

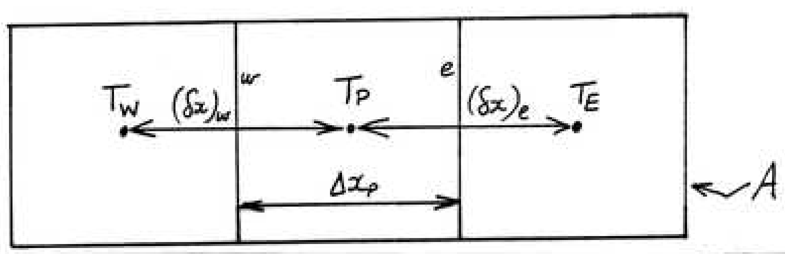
3.8 Given the following differential equation for T with positive constants a,b, and c

$$a\frac{d^2T}{dx^2} + b(c-T) = 0 \tag{P3-17}$$

Derive expressions for the coefficients a_P, a_W, a_E, and b_P of a discretization equation in the form

$$a_P T_P = a_W T_W + a_E T_E + b_P \tag{P3-18}$$

Use the nomenclature from Problem Figure 3.1 and briefly state any assumptions you make in the derivation. You may assume constant cross-sectional area.



Problem Figure 3.1: Nomenclature for Problem 3.8.

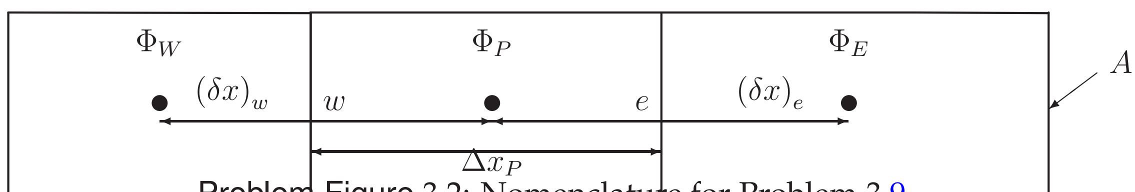
3.9 For the following differential equation for Φ with positive constant D

$$\frac{d^2\Phi}{dx^2} - D\,x^2\,\Phi = 0 \tag{P3-19}$$

integrate the differential equation for Φ over the typical control volume shown below, and derive expressions for the coefficients a_P, a_W, a_E, and b_P of a discretization equation in the form

$$a_P\Phi_P = a_W\Phi_W + a_E\Phi_E + b_P \tag{P3-20}$$

Use the nomenclature from Problem Figure 3.2 and clearly show the assumptions and approximations you make in the derivation. You may assume constant cross-sectional area and uniform grid spacing.



Problem Figure 3.2: Nomenclature for Problem 3.9.