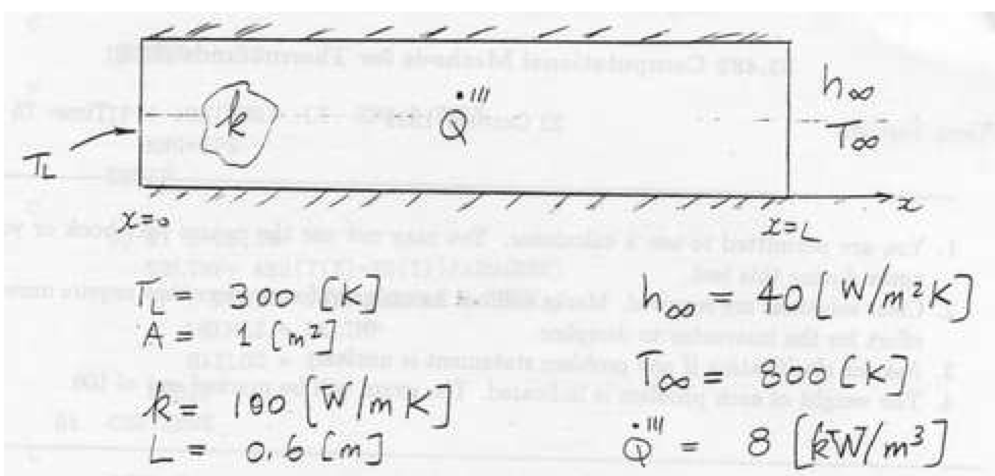
Problems

5.1 A computer program was run to model the one-dimensional problem of steady heat conduction in a bar with internal energy generation shown in Problem Figure 5.1.

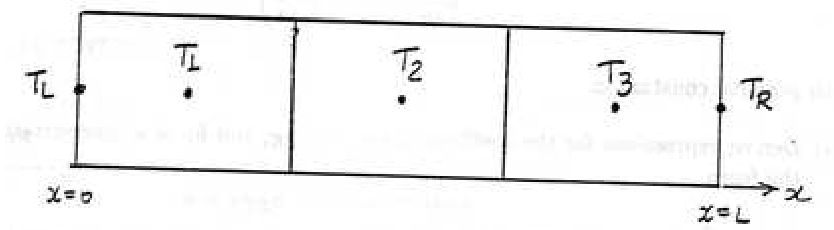


Problem Figure 5.1: Schematic for Problem 5.1.

The computer code used three control volumes with uniform grid spacing as shown in Problem Figure 5.2.

and produced the following results that you must check:

$T_1 \quad = 320.46\ [K]$
$T_2 \quad = 358.19\ [K]$
$T_3 \quad = 392.72\ [K]$



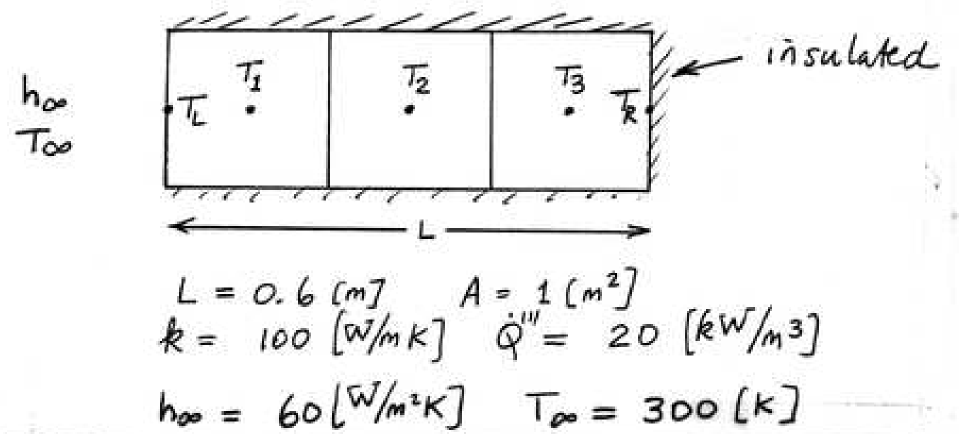
Problem Figure 5.2: Grid and node nomenclature for Problem 5.1.

(a) Using the solution provided by the computer code, demonstrate that energy is conserved in the control volumes for T_1 and T_2.

(b) Using energy conservation in the control volume for T_3, what value for T_R should the program have produced?

(c) According to an overall energy balance what should the value of T_R be?

5.2 A computer program was run to model the one-dimensional problem of steady heat conduction in a bar with internal energy generation using three control volumes with uniform grid spacing as shown schematically in Problem Figure 5.3.



Problem Figure 5.3: Schematic for Problem 5.2.

The computer code produced the following results that you must check:

$$T_1 = 512.0\ [K]$$
$$T_2 = 528.0\ [K]$$
$$T_3 = 536.0\ [K]$$

(a) By inspection, what should the value of T_R be?

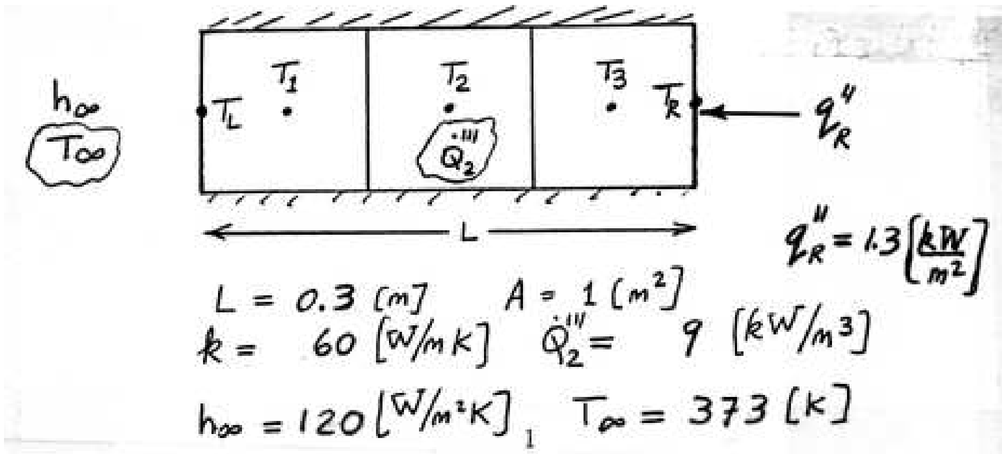
(b) Using the solution provided by the computer code, demonstrate that energy is conserved in the control volume for T_3.

(c) Using the solution provided by the computer code, demonstrate that energy is conserved in the control volume for T_2.

(d) Using energy conservation in the control volume for T_1, what value for T_L should the program have produced?

(e) According to an overall energy balance what should the value of T_L be?

5.3 A computer program was run to model the one-dimensional problem of steady heat conduction in a bar with internal energy generation (in **only** the control volume for T_2) using three control volumes with uniform grid spacing as shown in Problem Figure 5.4.



Problem Figure 5.4: Schematic for Problem 5.3.

The computer code produced the following results that you must check:

$T_1 = 393.17\ [K]$
$T_2 = 396.83\ [K]$
$T_3 = 399.00\ [K]$

(a) Using the solution provided by the computer code, demonstrate that energy is conserved in the control volume for T_3.

(b) Using the solution provided by the computer code, demonstrate that energy is conserved in the control volume for T_2.

(c) Using energy conservation in the control volume for T_1, what value for T_L should the program have produced?

(d) According to an overall energy balance what should the value of T_L be?

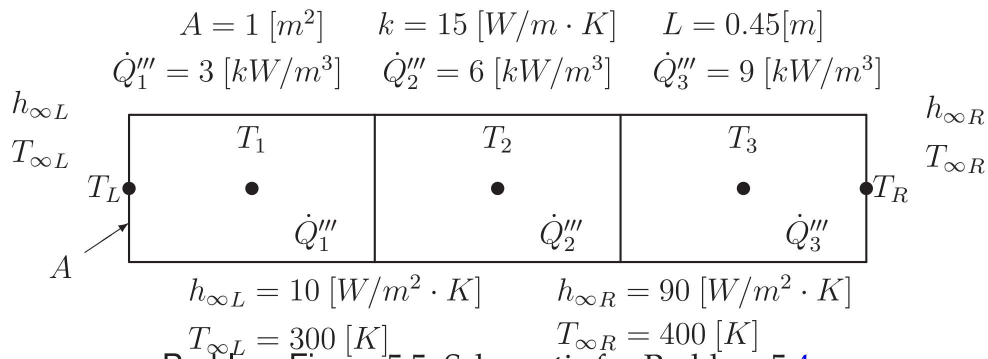
(e) What should the value of T_R be?

5.4 A computer program was run to model the one-dimensional problem of steady heat conduction in a bar with internal energy generation. The governing equation for this model was

$$k\,\frac{d^2T}{dx^2} + \dot{Q}''' = 0$$

The energy generation varies throughout the bar as specified in the diagram below. The program used three control volumes with uniform grid spacing as shown in Problem Figure 5.5.

The computer code produced the following results that you must check:



Problem Figure 5.5: Schematic for Problem 5.4.

$T_1 = 420.17\ [K] \qquad T_2 = 427.12\ [K] \qquad T_3 = 425.06\ [K]$

(a) Using the solution provided by the computer code, demonstrate that energy is conserved in the control volume for T_2.

(b) Using energy conservation in the control volume for T_1, what value for T_L should the program have produced?

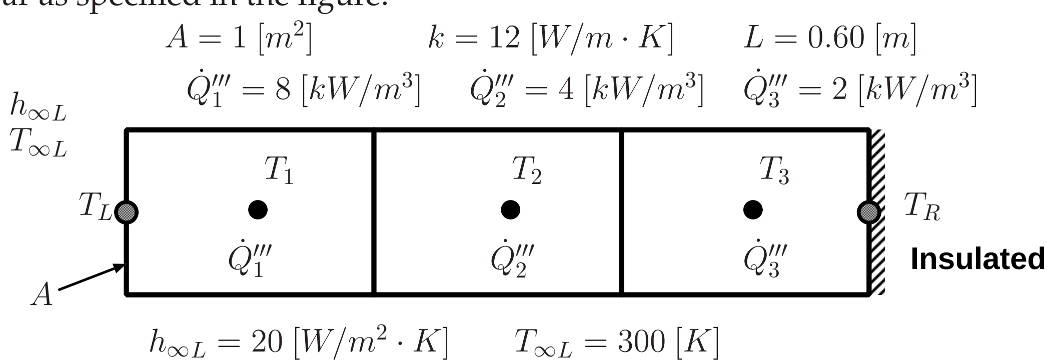
(c) Using energy conservation in the control volume for T_3, what value for T_R should the program have produced?

(d) Using the values of T_L and T_R calculated above, demonstrate an overall energy balance.

5.5 A computer program was run to model a one-dimensional problem of steady-state heat conduction in a bar with non-uniform internal energy generation. The governing equation for this model was:

$$k\,\frac{d^2T}{dx^2} + \dot{Q}''' = 0 \tag{P5-1}$$

The program used three control volumes with uniform grid spacing and the conditions shown in Problem Figure 5.6. The energy generation varies throughout the bar as specified in the figure.



Problem Figure 5.6: Schematic for Problem 5.5.

The computer code, based on the finite volume method described in this course, produced the following results:

$T_1 = \texttt{463.33}\ [K] \qquad T_2 = \texttt{483.33}\ [K] \qquad T_3 = \texttt{490.00}\ [K]$

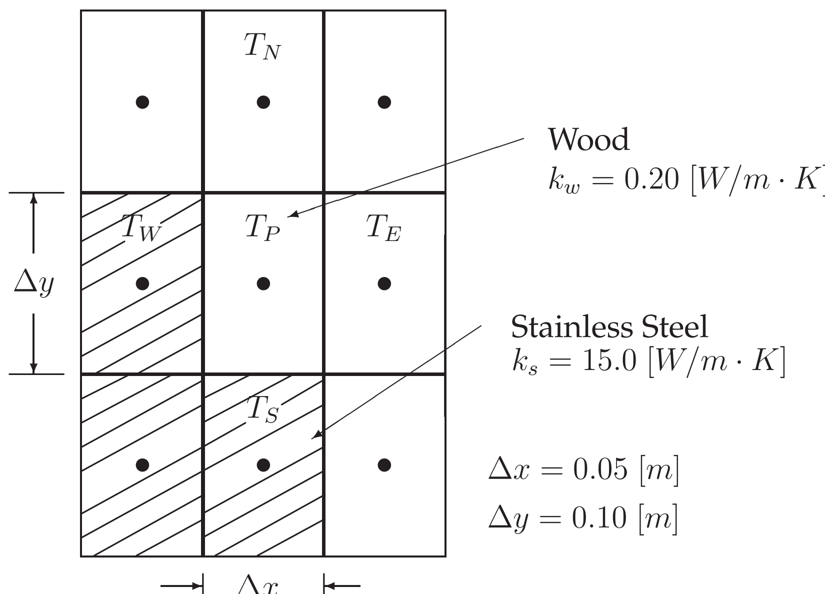
Perform the calculations below. Be consistent with the finite volume method.

(a) Using the solution provided by the computer code, demonstrate that energy is conserved in the control volume for T_2.

(b) Using energy conservation in the control volume for T_1, what value for T_L should the program have produced?

(c) Using energy conservation in the control volume for T_3, what value for T_R should the program have produced?

5.6 Problem Figure 5.7 shows the control volumes of a grid in a part of a larger domain in a composite solid material in which there is a temperature distribution due to two-dimensional steady conduction heat transfer. There are no volumetric energy generation sources.



Problem Figure 5.7: Schematic for Problem 5.6.

(a) Calculate the values of the coefficients a_P, a_W, a_E, a_S, a_N, and b_P of the algebraic equation in the form

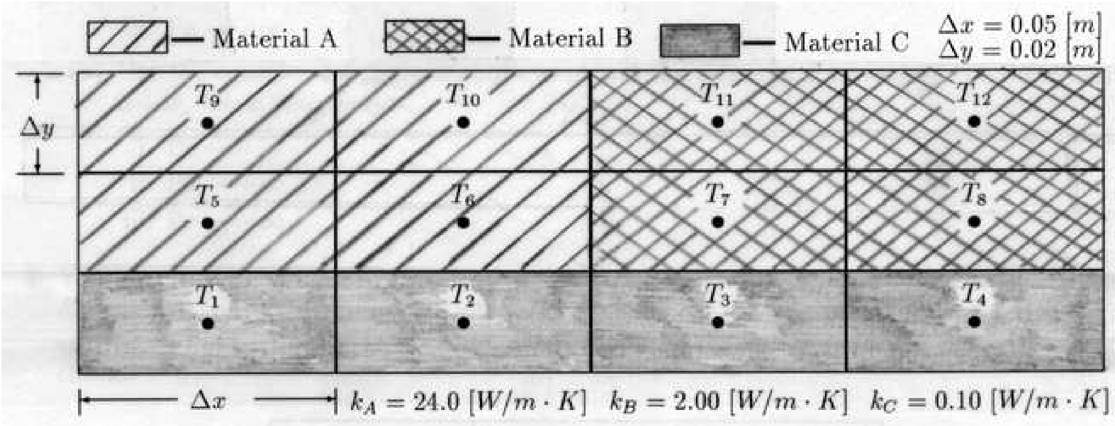
$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + b_P$$

Use the properties and the uniform grid spacing shown in Problem Figure 5.7.

(b) Using the algebraic equation and coefficients from part (a), calculate a "new iteration" value of T_P given the following values for the neighbouring nodal temperatures.

$T_W = 300\ [K]$ $\quad$ $T_E = 400\ [K]$ $\quad$ $T_S = 320\ [K]$ $\quad$ $T_N = 420\ [K]$

5.7 Problem Figure 5.8 shows the control volumes of a grid in a part of a larger domain in a composite solid material in which there is a temperature distribution due to



Problem Figure 5.8: Nomenclature for Problem 5.7.

two-dimensional steady conduction heat transfer. There are no volumetric energy generation sources.

In the questions below, use the properties and the uniform grid spacing shown in Problem Figure 5.8.

(a) Calculate the values of the coefficients a_P, a_W, a_E, a_S, a_N, and b_P of the algebraic equation in the form

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + b_P$$

for the control volume for T_6 (i.e. $T_P = T_6$).

(b) Calculate the values of the coefficients a_P, a_W, a_E, a_S, a_N, and b_P of the algebraic equation in the form

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + b_P$$

for the control volume for T_7 (i.e. $T_P = T_7$).

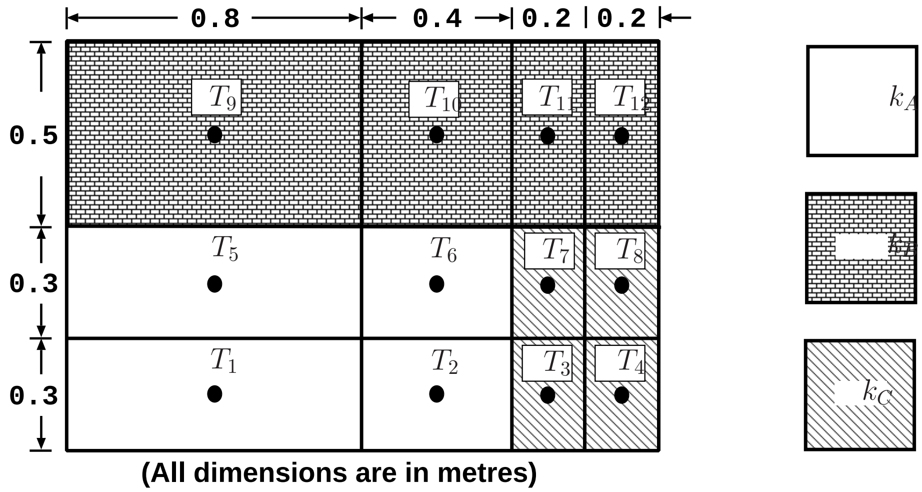
(c) Using the algebraic equations and coefficients from parts (a) and (b) and the old iteration values of the nodal temperatures given below:

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
360	400	500	550	320	320	380	500	300	320	360	400

form two equations for the two unknowns (new iteration values) T_6 and T_7. (Note that all other nodal values are held constant at the old iteration values).

(d) Solve for the new values of T_6 and T_7 simultaneously using the equations from part (c). Use substitution.

5.8 Problem Figure 5.9 shows the control volumes of a grid in a part of a larger domain in a composite solid material in which there is a temperature distribution due to two-dimensional steady-state conduction heat transfer. You may assume no internal energy generation and a unit depth.



Problem Figure 5.9: Composite material for Problem 5.8.

Use the grid dimensions shown in Problem Figure 5.9 and the grid-related data given in Problem Table 5.1. The values of the thermal conductivity for the three materials shown in the figure are $k_A = 16\ [W/m{\cdot}K]$, $k_B = 4\ [W/m{\cdot}K]$, and $k_C = 64\ [W/m{\cdot}K]$.

C.V.	$(\delta x)_e$	f_e	$(\delta x)_w$	f_w	$(\delta y)_n$	f_n	$(\delta y)_s$	f_s
T_6	0.30	0.33333	0.60	0.33333	0.40	0.62500	0.30	0.50000
T_7	0.20	0.50000	0.30	0.33333	0.40	0.62500	0.30	0.50000

Problem Table 5.1: Table of grid dimensions and weights for for Problem 5.8.

(a) Calculate the values of the coefficients a_P, a_W, a_E, a_S, a_N, and b_P of the algebraic equation in the form

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + b_P \tag{P5-2}$$

for the control volume for T_6 (i.e. $T_P = T_6$). Use the coefficients for the Finite Volume Method described in this course. You do not need to derive the equations for the coefficients.

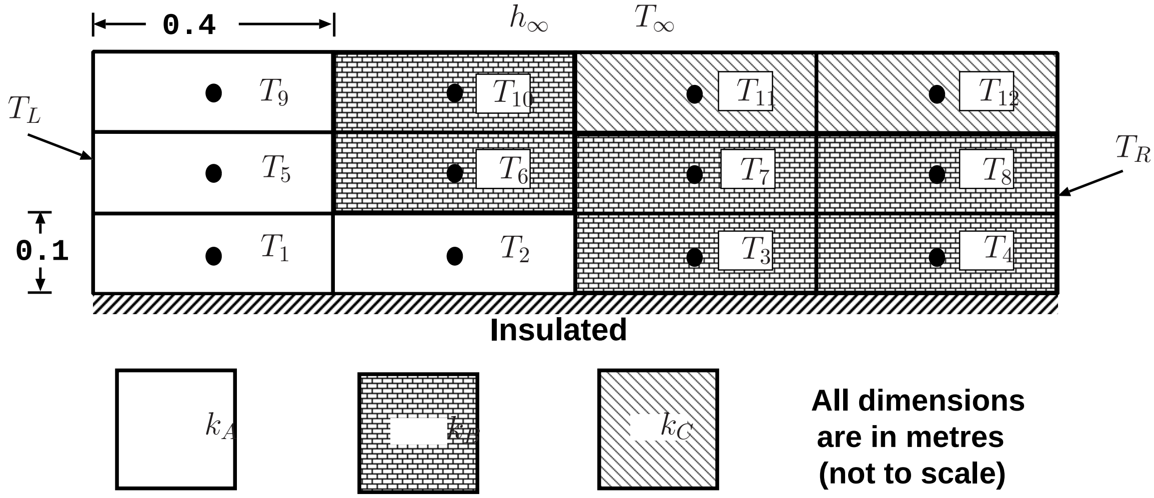
(b) Calculate the values of the coefficients a_P, a_W, a_E, a_S, a_N, and b_P of the algebraic equation in the form of Equation (P5-2) for the control volume for T_7 (i.e. $T_P = T_7$).

(c) Use the algebraic equations from parts (a) and (b) and the old iteration values of the nodal temperatures given below to form two equations for the two unknowns (new iteration values of) T_6 and T_7. For the purposes of this question hold all other nodal values constant at the old iteration values.

T_9	350 $[K]$	T_{10}	380 $[K]$	T_{11}	410 $[K]$	T_{12}	440 $[K]$
T_5	320 $[K]$	T_6	??	T_7	??	T_8	370 $[K]$
T_1	300 $[K]$	T_2	310 $[K]$	T_3	340 $[K]$	T_4	360 $[K]$

(d) Solve for the new values of T_6 and T_7 simultaneously using the equations from part (c).

5.9 Problem Figure 5.10 shows the control volumes of a uniformly-spaced grid in a composite solid material. In this domain, temperatures are specified as T_L=500 $[K]$ on the left boundary and T_R=300 $[K]$ on the right boundary; the bottom boundary is insulated; and the top boundary is exposed to convection heat transfer with h_∞=40 $[W/m^2 \cdot K]$ and T_∞=400 $[K]$. Assume two-dimensional steady-state conduction heat transfer with no internal energy generation and a unit depth.



Problem Figure 5.10: Composite material schematic for Problem 5.9.

The grid dimensions are shown in Problem Figure 5.10. The values of the thermal conductivity for the three materials shown in the figure are $k_A = 2$ $[W/m \cdot K]$, $k_B = 48$ $[W/m \cdot K]$, and $k_C = 12$ $[W/m \cdot K]$. Use the harmonic mean to calculate interface thermal conductivity as necessary.

(a) Calculate the values of the coefficients a_P, a_W, a_E, a_S, a_N, and b_P of the algebraic equation in the form

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + b_P \qquad \text{(P5-3)}$$

for the control volumes for T_1 and T_2. Use the coefficients for the Finite Volume Method described in this course. You do not need to use integration to derive the equations for the coefficients.

(b) Use the algebraic equations from part (a) and the known nodal temperatures given in Problem Table 5.2 to form two equations for the two unknowns (T_1 and T_2).

(c) Solve simultaneously for the values of T_1 and T_2 using the equations from part (c).

(d) Form the algebraic equation for T_{12} and determine its value.

T_9	416.12 $[K]$	T_{10}	383.16 $[K]$	T_{11}	366.69 $[K]$	T_{12}	??
T_5	425.70 $[K]$	T_6	382.06 $[K]$	T_7	360.79 $[K]$	T_8	330.28 $[K]$
T_1	??	T_2	??	T_3	359.00 $[K]$	T_4	328.60 $[K]$

Problem Table 5.2: Nodal values for Problem 5.9.

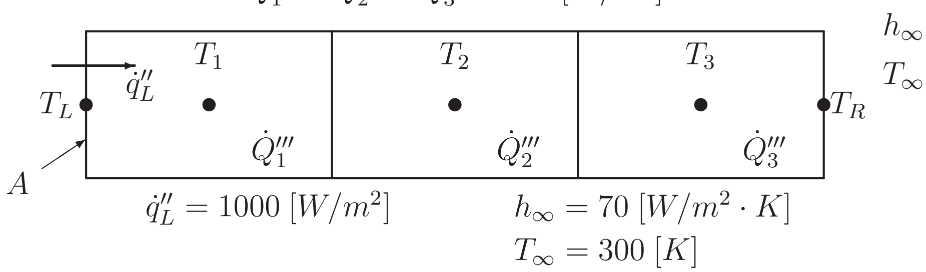
5.10 A finite volume discretization was used for one-dimensional steady conduction with a source term:

$$k\,\frac{d^2T}{dx^2} + \dot{Q}''' = 0$$

in the domain shown in Problem Figure 5.11. Note that in this problem, there is uniform grid spacing and the nodes to be used to prescribe the boundary conditions are *on the boundary*.

$$A = 1\ [m^2] \qquad k = 35\ [W/m \cdot K] \qquad L = 0.6 [m]$$

$$\dot{Q}_1''' = \dot{Q}_2''' = \dot{Q}_3''' = 555\ [W/m^3]$$



Problem Figure 5.11: Schematic for Problem 5.10.

For this problem, the algebraic equations for the three nodal temperatures **before** modification to include boundary conditions are:

$$525\,T_1 = 350\,T_L + 175\,T_2 + 111 \tag{P5-4}$$

$$350\,T_2 = 175\,T_1 + 175\,T_3 + 111 \tag{P5-5}$$

$$525\,T_3 = 175\,T_2 + 350\,T_R + 111 \tag{P5-6}$$

In order to prescribe the boundary equations given, the equations for the boundary nodes shown in the diagram are:

$$T_L = T_1 \; + \; \frac{\dot{q}_L''\,\Delta x}{2\,k} \tag{P5-7}$$

$$T_R = \frac{1}{\left(1 \; + \; \dfrac{h_\infty\,\Delta x}{2\,k}\right)}\,T_3 \; + \; \frac{\left(\dfrac{h_\infty\,\Delta x}{2\,k}\right)}{\left(1 \; + \; \dfrac{h_\infty\,\Delta x}{2\,k}\right)}\,T_\infty \tag{P5-8}$$

(a) Using the general boundary node equation for T_L given (Equation (P5-7) above), derive the boundary equation for T_L for this particular problem (i.e. substitute the values from the problem definition).

(b) Derive the modified algebraic equation for T_1 by absorbing the left boundary node equation derived in part (a).

(c) Using the general boundary node equation for T_R given (Equation (P5-8) above), derive the boundary equation for T_R for this particular problem (i.e. substitute the values from the problem definition).

(d) Derive the modified algebraic equation for T_3 by absorbing the right boundary node equation derived in part (c).

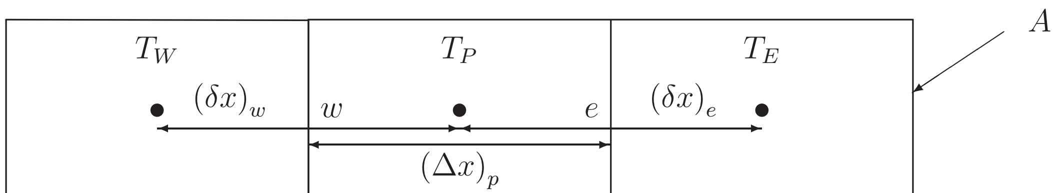
(e) Write out the new set of algebraic equations for T_1, T_2, and T_3 in matrix format. Solve the set of equations for T_1, T_2, and T_3.

5.11 A finite volume discretization method was used for one-dimensional steady conduction with a source term, for which the governing equation is:

$$\frac{d}{dx}\left(k\,\frac{dT}{dx}\right) + \dot{Q}''' = 0 \qquad \text{(P5-9)}$$

The governing equation was integrated over the typical control volume shown in Problem Figure 5.12 and the following discretization equation for a typical node was obtained:

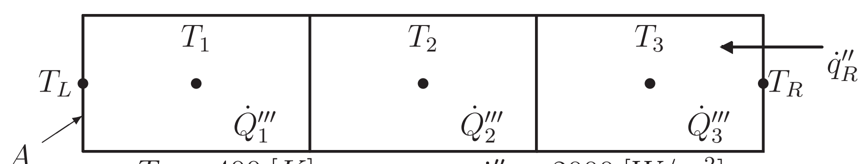
$$\left[\frac{k_w A_w}{(\delta x)_w} + \frac{k_e A_e}{(\delta x)_e}\right] T_P = \left[\frac{k_w A_w}{(\delta x)_w}\right] T_W + \left[\frac{k_e A_e}{(\delta x)_e}\right] T_E + \overline{\dot{Q}'''}(A)_p(\Delta x)_p \qquad \text{(P5-10)}$$



Problem Figure 5.12: Typical control volume for discretization for Problem 5.11.

The objective of this question is to derive and solve the set of algebraic equations needed to determine T_1, T_2, and T_3 for the problem shown in Problem Figure 5.13.

$$A = 1\ [m^2] \quad k = 70\ [W/m \cdot K] \quad L = 0.6[m]$$
$$\dot{Q}_1''' = \dot{Q}_2''' = \dot{Q}_3''' = 1110\ [W/m^3]$$



$$T_L = 400\ [K] \qquad \dot{q}_R'' = 2000\ [W/m^2]$$

Problem Figure 5.13: One-dimensional steady conduction with a source term problem for Problem 5.11.

(a) Apply Equation (P5-10) three times (once at each of the three control volumes: i.e. $T_P = T_1$, $T_P = T_2$, and $T_P = T_3$) to produce the set of 3 equations for T_1, T_2, and T_3.

Substitute properties, geometry, and source terms values as needed but leave T_L and T_R as symbols in the equations. Summarize the equation set in the form:

$$[a_P]_1\, T_1 = [a_W]_1\, T_L + [a_E]_1\, T_2 + [b_P]_1$$

$$[a_P]_2\, T_2 = [a_W]_2\, T_1 + [a_E]_2\, T_3 + [b_P]_2$$
$$[a_P]_3\, T_3 = [a_W]_3\, T_2 + [a_E]_3\, T_R + [b_P]_3$$

where the $[a_P]$,$[a_W]$, $[a_E]$, and $[b_P]$ coefficients will have been calculated using the problem definition values.

The value of T_L will be substituted later and an equation for T_R will be derived next.

(b) The equation for T_R (from the boundary condition) is:

$$T_R = T_3 \; + \; \frac{\dot{q}''_R\, \Delta x}{2\, k} \tag{P5-11}$$

Substitute values for $\dot{q}''_R$, Δx, and k to obtain the specific equation for T_R for the problem in Problem Figure 5.13.

(c) Substitute for T_L and T_R (using the known value for T_L from the boundary condition and the equation from part (b) for T_R, respectively) into your set of equations from part (a). These will then be the modified algebraic equations after absorbing the boundary conditions. Solve the set of equations for T_1, T_2, and T_3.

(d) Using the solution from part (c), demonstrate that your discrete (numerical) solution obeys overall energy conservation for the domain defined in Problem Figure 5.13.

5.12 This question examines transient two-dimensional conduction with no source term. The governing differential equation is:

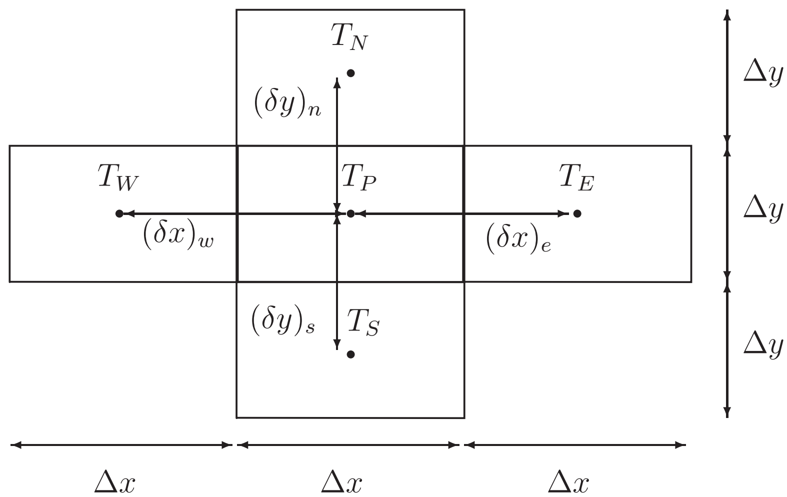
$$\frac{\partial}{\partial t}(\rho\, T) \; = \; \frac{\partial}{\partial x}\left(\Gamma\, \frac{\partial T}{\partial x}\right) \; + \; \frac{\partial}{\partial y}\left(\Gamma\, \frac{\partial T}{\partial y}\right)$$

where $\Gamma = \dfrac{k}{C_p}$, k is the thermal conductivity, ρ is the density, and C_p is the specific heat.

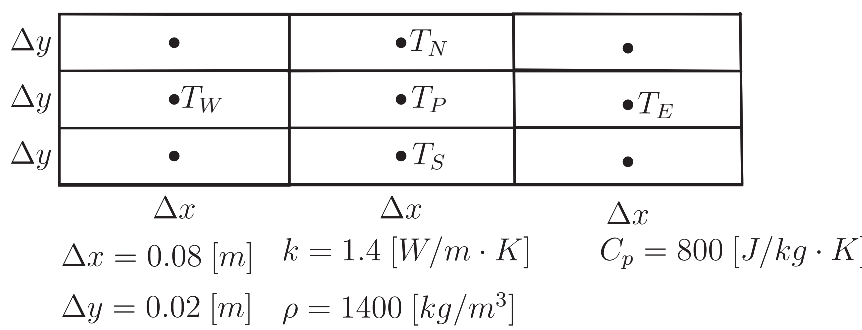
(a) Using the finite volume approach presented in this course, integrate the differential equation over time and space to derive expressions for the coefficients a_P, a_E, a_S, a_N, a_S, and b_P of a discretization equation in the form

$$a_P T_P^n = a_E T_E^n + a_W T_W^n + a_N T_N^n + a_S T_S^n + b_P$$

for the Fully Explicit scheme only (corresponding to $f_t = 0$). Use the nomenclature in Problem Figure 5.14, use T^n and T^o to indicate "new" and "old" time step values of T, and include the assumptions and approximations you make in the derivation. You may assume uniform grid spacing and unit depth, but you may <u>not</u> assume that Δx and Δy are equal. Note that you do not need to do the general time-weighting derivation; you may derive the Fully Explicit scheme directly.



Problem Figure 5.14: Typical control volume for discretization for Problem 5.12.



Problem Figure 5.15: Two-dimensional unsteady conduction with no source term problem for Problem 5.12.

(b) Now, consider the domain shown in Problem Figure 5.15.

Using the discretization equation from part (a), substitute the relevant values from the domain defined above and write out the algebraic equation for T_P^n. Leave the time step, Δt, and the nodal temperatures from the previous time step as variables in this equation.

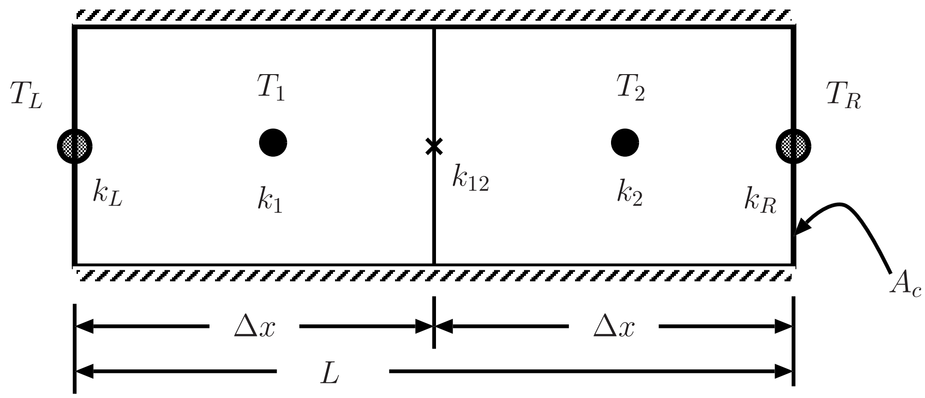
(c) The values of the nodal temperatures at and around T_P from the previous time step are known to be

$$T_P^o = 375\ [K] \qquad T_W^o = 400\ [K] \qquad T_E^o = 450\ [K] \qquad T_S^o = 350\ [K] \qquad T_N^o = 300\ [K]$$

Using the algebraic equation from part (b), calculate the value of T_P^n after one time step (Δt), for Δt = 2 minutes.

(d) What is the maximum time step that may be used in the calculation of T_P^n for this specific problem?

5.13 A bar made of AISI 304 stainless steel of length L is insulated along its length and held at T_L on one end and T_R on the opposite end as shown in Problem Figure 5.16. The cross-sectional area of the rod is A_c and there are no internal energy sources.



Problem Figure 5.16: AISI 304 Stainless steel bar schematic for Problem 5.13.

The objective of the analysis in this problem is to formulate and to perform a numerical solution for the temperature distribution (using two control volumes) including the effect of thermal conductivity variation with temperature. The case of constant thermal conductivity will be solved and used as a basis for comparison.

A correlation for the thermal conductivity variation with temperature of AISI 304 stainless steel is

$$k\,(T) \;=\; C_k\, T^m \qquad \text{(P5-12)}$$

where

$$C_k \;=\; 1.2073 \qquad \text{(P5-13)}$$

$$m \;=\; 0.441 \qquad \text{(P5-14)}$$

k is in $[W/m \cdot K]$, and T is in absolute temperature. The correlation is valid for the range: 100 $[K] \leq T \leq$ 1000 $[K]$.

(a) For this steady, one-dimensional heat conduction problem, show that the algebraic equations for T_1 and T_2 can be simplified, in this case, to:

$$(2\,k_L \;+\; k_{12})\;T_1 \;-\; (k_{12})\;T_2 \;=\; (2\,k_L)\;T_L \qquad \text{(P5-15)}$$

$$-\;(k_{12})\;T_1 \;+\; (2\,k_R \;+ k_{12})\;T_2 \;=\; (2\,k_R)\;T_R \qquad \text{(P5-16)}$$

where k_L, k_R, k_1, k_2, and k_{12} are the values of the thermal conductivity at the left boundary, the right boundary, the T_1 node, the T_2 node, and the interface between the T_1 and T_2 control volumes, respectively.

(b) For the values of L = 1.87 $[m]$, A_c = 0.05 $[m^2]$, T_L = 100 $[K]$, and T_R = 900 $[K]$, solve for T_1 and T_2 using the equations from part (a) (Equations (P5-15) and (P5-16)) for the case of constant thermal conductivity. In this case, use a value of 18.7 $[W/m \cdot K]$ for all k values (i.e. k_L=k_R=k_1=k_2=k_{12}=18.7 $[W/m \cdot K]$).

(c) Compare the T_1 and T_2 values calculated in part (b) with the analytical solution for $T(x)$ for that case.

(d) Now set up the equations to solve the non-linear problem when k varies with temperature as given in Equation (P5-12). In this case use:

$$k_L = k_1 = k(T_1) \tag{P5-17}$$

$$k_R = k_2 = k(T_2) \tag{P5-18}$$

and the harmonic mean to calculate k_{12}. Show that the linearized equations for T_1 and T_2 are, in this case:

$$\left[(T_1^*)^m + \frac{(T_1^*)^m (T_2^*)^m}{\{(T_1^*)^m + (T_2^*)^m\}}\right] T_1 - \left[\frac{(T_1^*)^m (T_2^*)^m}{\{(T_1^*)^m + (T_2^*)^m\}}\right] T_2 = (T_1^*)^m T_L \tag{P5-19}$$

$$-\left[\frac{(T_1^*)^m (T_2^*)^m}{\{(T_1^*)^m + (T_2^*)^m\}}\right] T_1 + \left[(T_2^*)^m + \frac{(T_1^*)^m (T_2^*)^m}{\{(T_1^*)^m + (T_2^*)^m\}}\right] T_2 = (T_2^*)^m T_R \tag{P5-20}$$

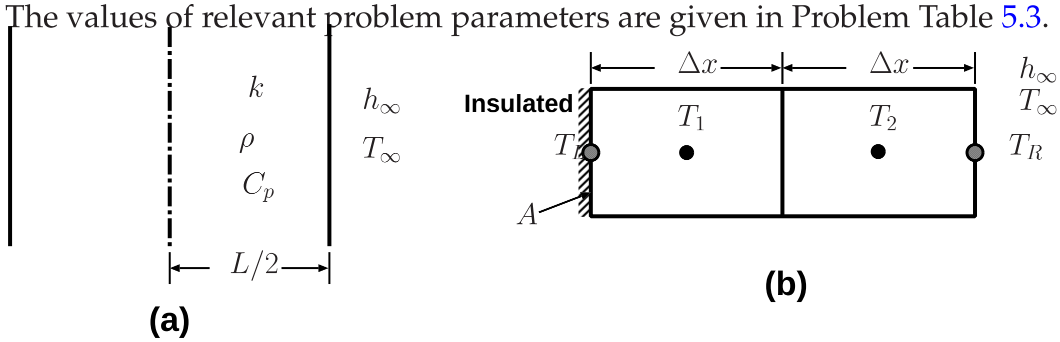
where T_1^* and T_2^* are the previous (or guessed) values of T_1 and T_2.

(e) Solve for T_1 and T_2 using Equations (P5-19) and (P5-20). Use the solution from part (b) as the intial guess for T_1 and T_2. In the solution procedure, do a direct solution of the linearized equations for T_1 and T_2 for each iteration. Do two full iterations (i.e. two solutions of the linearized equations).

5.14 The transient temperature field in the plane wall shown in Problem Figure 5.17(a) is to be computed using the Finite Volume method described in this course. Using symmetry, half the domain is modelled with two control volumes as shown in Problem Figure 5.17(b). The governing equation that applies in this case is:

$$\frac{\partial}{\partial t}(\rho T) = \frac{\partial}{\partial x}\left(\frac{k}{C_p}\frac{\partial T}{\partial x}\right) \tag{P5-21}$$

The values of relevant problem parameters are given in Problem Table 5.3.



Problem Figure 5.17: Problem definition for Problem 5.14.

This question requires that the appropriate algebraic equations for T_1^n and T_2^n be determined for both the Fully Implicit and the Fully Explicit approaches, and that calculations of the nodal temperatures be made at 3 time steps for both approaches.

Note that for the first time step $T_1^o = T_2^o = T_L^o = T_R^o = T_i$, where T_i is the uniform initial temperature at time zero. Note also for parts (c) and (e) below that in the control volume for T_1: $T_W = T_L$ and $(\delta x)_w = \Delta x/2$, and that in the control volume for T_2:

Parameter	Value	Parameter	Value
k	$58.5\ [W/m \cdot K]$	h_∞	$500\ [W/m^2 \cdot K]$
ρ	$7800\ [kg/m^3]$	T_∞	$333\ [K]$
C_p	$390\ [J/kg \cdot K]$	T_i	$253\ [K]$
L	$0.06\ [m]$		
A	$1\ [m^2]$	Δt	$30\ [s]$
Δx	$0.015\ [m]$		

Problem Table 5.3: Table of parameter values for Problem 5.14.

$T_E = T_R$ and $(\delta x)_e = \Delta x/2$. Finally, in all parts below that require calculations, keep 5 significant figures.

(a) Show that the boundary condition on the left leads to the following equation:

$$T_L = T_1 \tag{P5-22}$$

(b) Show that the boundary condition on the right leads to the following equation:

$$T_R = \frac{2}{(2+\text{Bi})} T_2 + \frac{\text{Bi}}{(2+\text{Bi})} T_\infty \tag{P5-23}$$

where $\text{Bi} = \dfrac{h_\infty \Delta x}{k}$.

(c) Starting from Equations (5.124) to (5.126), using the conditions given in Table 5.3, and **absorbing the boundary condition equations**, show that the equations for T_1^n and T_2^n for the **Fully Implicit** approach are Equations (P5-24) and (P5-25) below. You may begin with an already simplified version of the starting equations.

$$13.9\, T_1^n = 10\, T_2^n + 3.9\, T_1^o \tag{P5-24}$$

$$15.105\, T_2^n = 10\, T_1^n + 3.9\, T_2^o + 401.20 \tag{P5-25}$$

(d) Perform the calculation of T_1^n and T_2^n for 3 time steps using a simultaneous solution of Equations (P5-24) and (P5-25) (i.e. calculate T_1 and T_2 at times of 30, 60, and 90 seconds using the **Fully Implicit** approach). To perform the simultaneous solution, you may use Equations (P5-26) and (P5-27) below (which come from substitution of Equation (P5-24) into (P5-25) and re-writing Equation (P5-24)).

$$T_2^n = \frac{(39\, T_1^o + 54.210\, T_2^o + 5576.7)}{109.96} \tag{P5-26}$$

$$T_1^n = \frac{(10\, T_2^n + 3.9\, T_1^o)}{13.9} \tag{P5-27}$$

(e) Starting from Equations (5.124) to (5.126), using the conditions given in Problem Table 5.3, and **leaving the boundary nodes explicit in the nodal temperature equations** (i.e. do not absorb them), show that the equations for T_1^n and T_2^n for the **Fully Explicit** approach are Equations (P5-28) and (P5-29) below.

$$3.9\, T_1^n = 20\, T_L^o + 10\, T_2^o - 26.1\, T_1^o \tag{P5-28}$$

$$3.9\,T_2^n \;=\; 10\,T_1^o \;+\; 20\,T_R^o - \;26.1\,T_2^o \tag{P5-29}$$

(f) Perform the calculation of T_1^n and T_2^n for 3 time steps using Equations (P5-28) and (P5-29) (i.e. calculate T_1 and T_2 at times of 30, 60, and 90 seconds using the **Fully Explicit** approach). Make sure to update the boundary node values **after** the calculation of T_1^n and T_2^n but **before** the next time step. The boundary node equations are:

$$T_L^n \;=\; T_1^n \tag{P5-30}$$

$$T_R^n \;=\; 0.93976\,T_2^n \;+\; 20.060 \tag{P5-31}$$

(g) Explain the difference between the values calculated for T_1 and T_2 at a time of 90 seconds in parts (d) and (f).

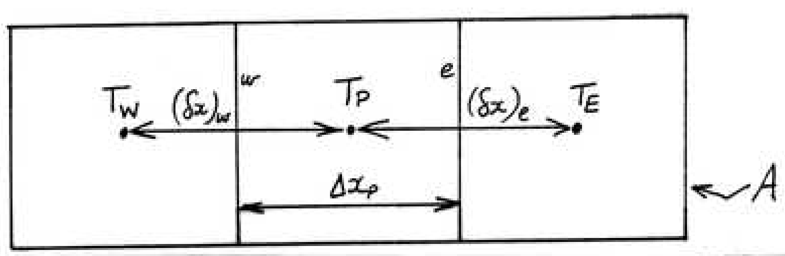
5.15 Given the following differential equation for T with positive constant a:

$$a\frac{d^2T}{dx^2} + S = 0$$

(a) By performing the integration of the differential equation over a typical control volume shown in Problem Figure 5.18 and using the nomenclature given, derive expressions for the coefficients a_P, a_W, a_E, and b_P of a discretization equation in the form

$$a_P T_P = a_W T_W + a_E T_E + b_P$$

You may assume constant cross-sectional area, uniform grid spacing and constant source term S.



Problem Figure 5.18: Nomenclature for Problem 5.15.

(b) Now examine the case when S is a volumetric energy source term which is a function of T given by

$$S = b\,(T_\infty^4 - T^4)$$

where b and T_∞ are positive constants.
Derive the coefficients S_c and S_p for the linearization of S in the form:

$$S = S_c + S_p T_P$$

using the Newton-Raphson linearization

$$S = S^* + \left.\frac{\partial S}{\partial T}\right|_{T_P^*}(T_P - T_P^*)$$

Use this new linearization to redo the integration of S. Use the results of the new integration of S to derive the new coefficients of the general discretization equation.

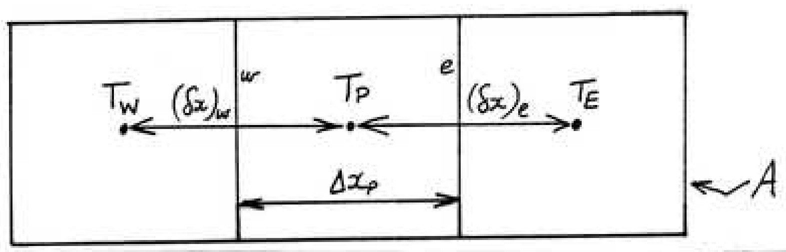
5.16 Given the following differential equation for T with positive constant a:

$$a\frac{d^2T}{dx^2} + S = 0$$

(a) By performing the integration of the differential equation over a typical control volume shown in Problem Figure 5.19 and using the nomenclature given, derive expressions for the coefficients a_P, a_W, a_E, and b_P of a discretization equation in the form

$$a_P T_P = a_W T_W + a_E T_E + b_P$$

You may assume constant cross-sectional area, uniform grid spacing and constant source term S.



Problem Figure 5.19: Nomenclature for Problem 5.16.

(b) Now examine the case when S is a volumetric energy source term which is a function of T given by

$$S = -b\left(T - T_\infty\right)^3$$

where b and T_∞ are positive constants.

Derive the coefficients S_c and S_p for the linearization of S in the form:

$$S = S_c + S_p T_P$$

using the Newton-Raphson linearization

$$S = S^* + \left.\frac{\partial S}{\partial T}\right|_{T_P^*}(T_P - T_P^*)$$

Use this new linearization to redo the integration of S. Use the results of the new integration of S to derive the new coefficients of the general discretization equation.

5.17 Given the following differential equation for T:

$$\frac{d^2T}{dx^2} + \dot{S}''' = 0$$

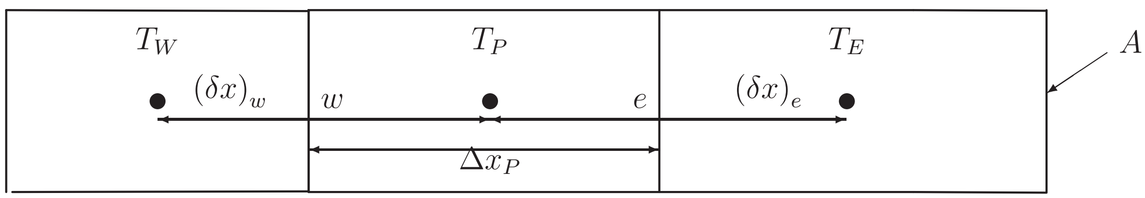
with a source term that is a non-linear function of T. The source term is linearized using the following equation:

$$\dot{S}''' = S_c + S_p T$$

(a) By performing the integration of the differential equation over a typical control volume shown in Problem Figure 5.20 and using the nomenclature given, derive expressions for the coefficients a_P, a_W, a_E, and b_P of a discretization equation in the form:

$$a_P T_P = a_W T_W + a_E T_E + b_P$$

You may assume constant cross-sectional area, uniform grid spacing, and a step-wise profile when integrating the source term.



Problem Figure 5.20: Schematic for Problem 5.17.

(b) Now examine the specific case when $\dot{S}'''$ is a volumetric energy source term which is a function of T given by

$$\dot{S}''' = -C\,(T - T_\infty)^{\frac{1}{4}}$$

where C and T_∞ are positive constants.
Using the Newton-Raphson linearization:

$$(\dot{S}''')_p = (\dot{S}''')_p^* + \left.\frac{\partial(\dot{S}''')}{\partial T}\right|_{T_P^*} (T_P - T_P^*)$$

derive expressions for the S_c and S_p coefficients in the equation

$$(\dot{S}''')_p = S_c + S_p T_P$$

Note that $(\dot{S}''')_p^*$ is the source term function evaluated using T_P^*, which is the previous guessed value of T_P.

5.18 For each of the non-linear source term functions given below, derive expressions for the S_c and S_p coefficients in the linearization equation:

$$(S''')_p = S_c + S_p T_P$$

Use the Newton-Raphson linearization:

$$\left(\dot{S}'''\right)_p = \left(\dot{S}'''\right)_p^* + \left.\frac{\partial(\dot{S}''')}{\partial T}\right|_{T_P^*} (T_P - T_P^*)$$

where $\left(\dot{S}'''\right)_p^*$ is the source term function evaluated using T_P^*, which is the previous guessed value of T_P.

(a) For positive constants C and T_∞:

$$\dot{S}''' = -\,C\,(T - T_\infty)^{\frac{1}{4}}$$

(b) For positive constants K and T_{surr}:

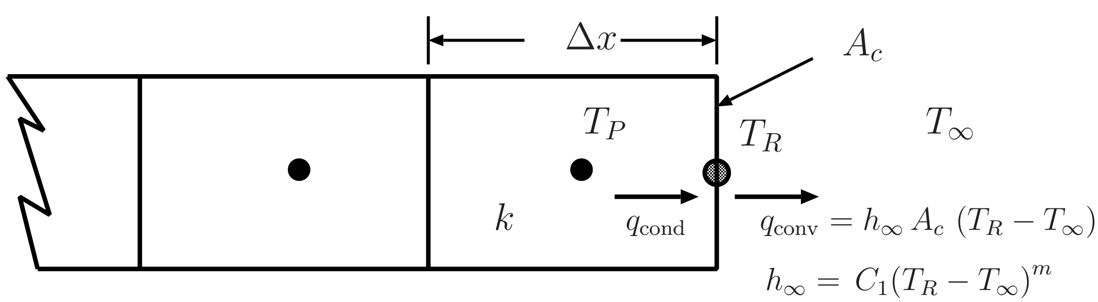
$$\dot{S}''' = -\,K\,\left(T^4 - {T_{surr}}^4\right)$$

5.19 The objective of this analysis is to derive a general formulation for a non-linear convection heat transfer boundary condition for a boundary node in a Finite Volume Method discretization. Note that the convection heat transfer coefficient at the boundary, h_∞, depends on the temperature difference, $(T_R - T_\infty)$, raised to the power m.

For the typical east boundary control volume shown in Problem Figure 5.21, derive the coefficients a_{PR}, a_{WR}, and b_{PR} for the boundary node equation given in Equation (P5-32).

$$a_{PR} T_R = a_{WR} T_P + b_{PR} \tag{P5-32}$$

You may assume that T_R is always a node on the boundary. Note that the boundary node equation must be <u>linear</u> in T_P and T_R. Therefore, in your derivation, use a Newton-Raphson linearization of the expression for the convection heat transfer at the boundary. Use the notation that T_R^* is the previous iteration (or guessed) value of T_R. Show your work.



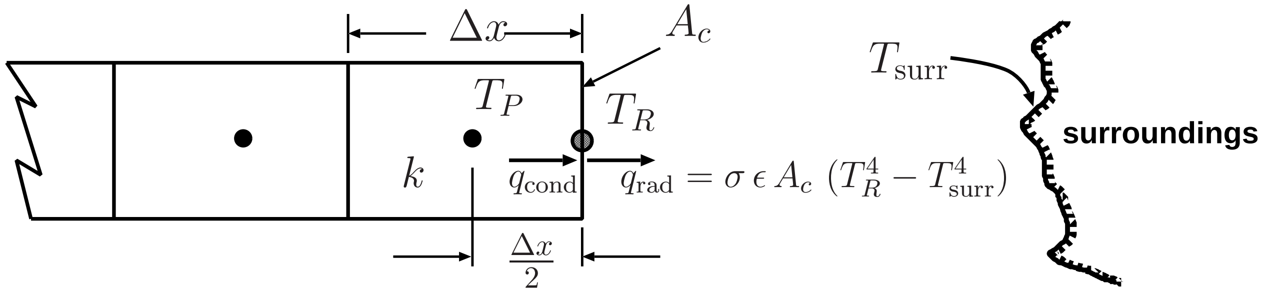
Problem Figure 5.21: Convection boundary condition schematic for Problem 5.19.

5.20 Problem Figure 5.22 shows a radiation heat transfer boundary condition on the right boundary of a typical domain.

For the typical east boundary control volume shown in the figure, derive expressions for the coefficients a_{PR}, a_{WR}, and b_{PR} for the boundary node equation given in Equation (P5-33).

$$a_{PR}T_R = a_{WR}T_P + b_{PR} \tag{P5-33}$$

You may assume that T_R is always a node on the boundary. Note that the boundary node equation must be *linear* in T_P and T_R. Therefore, in your derivation, use a Newton-Raphson linearization of the expression for the heat transfer at the boundary. Use the notation that T_R^* is the previous iteration (or guessed) value of T_R.



Problem Figure 5.22: Radiation boundary condition schematic for Problem 5.20.

5.21 For the following non-linear differential equation for ϕ

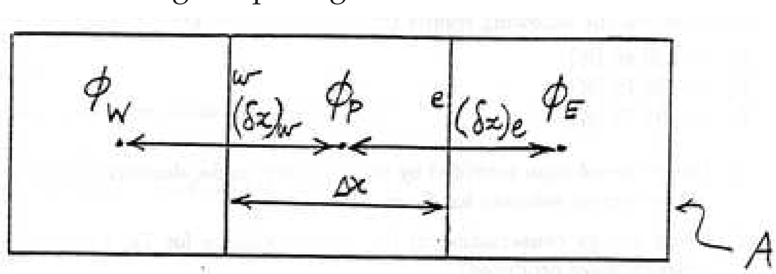
$$c\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left(\phi\frac{\partial \phi}{\partial x}\right) \tag{P5-34}$$

with positive constant c:

(a) Derive expressions for the coefficients a_P, a_W, a_E, and b_P of a discretization equation in the form

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + b_P \tag{P5-35}$$

for the general case using a time weighting factor f_t. Use the nomenclature from Problem Figure 5.23 and clearly show the assumptions and approximations you make in the derivation. You may assume constant cross-sectional area and uniform grid spacing.



Problem Figure 5.23: Nomenclature for Problem 5.21.

(b) Write out the general discretization equation for the Explicit time weighting.

(c) What is the stability criterion for the Explicit equation from part (b)?

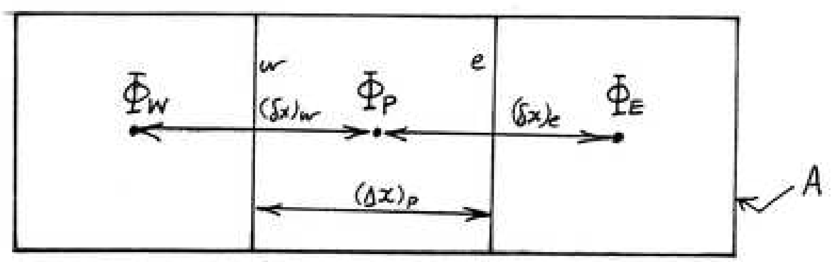
5.22 For the following differential equation for Φ with positive constants a, b, and c:

$$\frac{\partial \Phi}{\partial t} = a\,(b - \Phi) + c\,\frac{\partial \Phi}{\partial x} \tag{P5-36}$$

(a) By integrating over time and space, derive expressions for the coefficients a_P, a_W, a_E, and b_P of a discretization equation in the form

$$a_P \Phi_P^n = a_W \Phi_W^n + a_E \Phi_E^n + b_P \tag{P5-37}$$

for the general case using a time weighting factor f_t. Use the nomenclature from Problem Figure 5.24 and clearly show the assumptions and approximations you make in the derivation. You may assume constant cross-sectional area and uniform grid spacing.



Problem Figure 5.24: Nomenclature for Problem 5.22.

(b) Write out the general discretization equation for the Explicit ($f_t = 0$) time weighting.

5.23 For the following differential equation for Φ with positive constants a and b:

$$\frac{\partial \Phi}{\partial t} = a\,\frac{\partial \Phi}{\partial x} - b\,x\,\Phi \tag{P5-38}$$

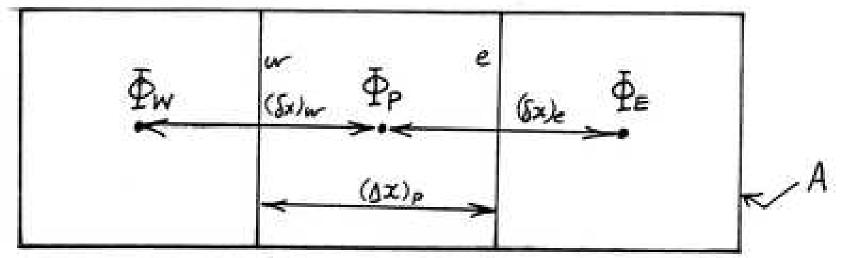
(a) By integrating over time and space, derive expressions for the coefficients a_P, a_W, a_E, and b_P of a discretization equation in the form

$$a_P \Phi_P^n = a_W \Phi_W^n + a_E \Phi_E^n + b_P \tag{P5-39}$$

for the general case using a time weighting factor f_t. Use the nomenclature from Problem Figure 5.25 and clearly show the assumptions and approximations you make in the derivation. You may assume constant cross-sectional area and uniform grid spacing.

(b) Write out the general discretization equation for the Fully Implicit ($f_t = 1$) scheme.

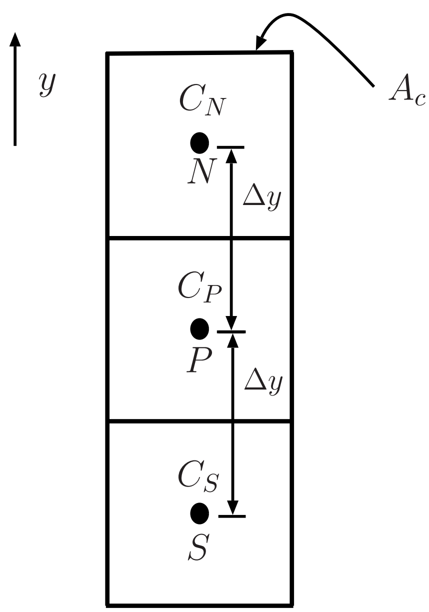
5.24 In a particular application, the time-varying concentration, C, of a substance in a vertical channel is governed by Equation (P5-40).



Problem Figure 5.25: Nomenclature for Problem 5.23.

$$\frac{\partial C}{\partial t} = \Gamma \frac{\partial^2 C}{\partial y^2} - \beta (C - 1) \tag{P5-40}$$

where Γ and β are positive constants.



Problem Figure 5.26: Typical control volume at P for Problem 5.24.

(a) For the typical control volume centred at the P point shown in Problem Figure 5.26, derive the coefficients a_P, a_N, a_S, and b_P for the discretization equation given in Equation (P5-41)

$$a_P C_P^n = a_N C_N^n + a_S C_S^n + b_P \tag{P5-41}$$

where the superscript "n" refers to a new time step value of a quantity. Use the finite volume method discussed in this course with a general time weight factor, f_t, constant cross-sectional area, A_c, uniform grid spacing and refer to an old time step value of a quantity with a superscript "o". Note that in this question C_P is the P point value of C and not the specific heat. State your assumptions. Note: in the time integration of the second term on the right hand side of Equation (P5-40), use only the new time step value.

(b) Write out the discretization equation for the Fully Explicit case (neighbour values taken at the old time step level). Is there a time step restriction? If so, derive an expression for it in terms of the appropriate problem parameters.

(c) Using the Fully Implicit version of the discretization equation, derive the algebraic equation for C_P that would be used to calculate the steady-state C field.