Convexity
(section 7.5)

The concept of duration was studied when we used a linear approximation for $P(y + \Delta y)$. To achieve a more accurate approximation one needs to employ higher order derivatives in the Taylor expansion. If we stop at the second order derivative, we will have the approximation:

$$P(y_0 + \Delta y) \approx P(y_0) + P'(y_0)(\Delta y) + \frac{1}{2} P''(y_0)(\Delta y)^2$$

Then:

$$\frac{P(y_0 + \Delta y) - P(y_0)}{P(y_0)} \approx (\Delta y) \frac{P'(y_0)}{P(y_0)} + \frac{(\Delta y)^2}{2} \frac{P''(y_0)}{P(y_0)}$$

$$\% \Delta P \approx (\Delta y) \frac{P'(y_0)}{P(y_0)} + \frac{(\Delta y)^2}{2} \frac{P''(y_0)}{P(y_0)}$$

**Definition.** The term $\frac{P''(y_0)}{P(y_0)}$ is called the *convexity* at $y_0$.

So:

$$\% \Delta P \approx -(\Delta y)\text{(Duration)} + \frac{(\Delta y)^2}{2}\text{Convexity}$$

**Note.** The following formula can be used to calculate convexity:

$$v^n = (1 + y)^{-n} \Rightarrow \frac{d(v^n)}{dy} = -n(1 + y)^{-n-1} = -n v^{n+1}$$

$$\frac{d(v^n)}{dy} = -n v^{n+1}$$

**Note.** For bonds with fixed cash flows, convexity is positive for all $y_0$’s.

**Example (from the study manual).** A $1000 3$-year par value bond yields an effective annual interest rate of 6%. Coupons are paid an annual basis at a rate of 5% per year.
Determine the convexity of the bond.

**Solution.**

\[ P = 50 a_{3\%} + 1000 v^3 = 973.27 \]

\[ P(y) = 50(v + v^2) + 1050 v^3 \]

\[ P' = 50(-v^2 - 2v^3) + 1050(-3v^4) \]

\[ P'' = 50(2v^3 + 6v^4) + 1050(12v^5) = \text{at } 6\% = 9,737.04 \]

Convexity = \[ \frac{P''}{P} = \frac{9,737.04}{973.27} = 10.00 \]

**Note.** Macaulay convexity (MacC), which is \( \frac{d^2P}{d^2\delta} \), has a simpler formula and is more widely used. In fact:

\[ P = \sum_{t>0} CF_t e^{-\delta t} \quad \Rightarrow \quad \frac{dP}{d\delta} = \sum_{t>0} (-t)CF_t e^{-\delta t} \quad \Rightarrow \quad \frac{d^2P}{d\delta^2} = \sum_{t>0} t^2CF_t e^{-\delta t} \]

Dividing it by \( P \) we get the Macaulay Duration:

**Macaulay Convexity:**

If a bond with fixed cash flows has a continuously compounded yield of \( \delta \), then its Macaulay Convexity is:

\[ MacC = \frac{\sum_{t>0} t^2CF_t e^{-\delta t}}{P} \]

**Definition.** Dispersion is defined as the following weighted average:
Dispersion = \frac{\sum_{t>0} (t - MacD)^2 CF_t e^{-\delta t}}{\sum_{t>0} CF_t e^{-\delta t}}

\textbf{Note.}

Dispersion = \frac{\sum_{t>0} \left\{ t^2 - 2t MacD + MacD^2 \right\} CF_t e^{-\delta t}}{P}

= \frac{\sum_{t>0} t^2 CF_t e^{-\delta t} - 2MacD \sum_{t>0} t CF_t e^{-\delta t} + MacD^2 \sum_{t>0} CF_t e^{-\delta t}}{P}

= MacC - 2MacD^2 + MacD^2

= MacC - MacD^2

\textbf{Dispersion:}

If a bond with fixed cash flows has a continuously compounded yield of \( \delta \), then its Macaulay Convexity can be written as:

\( MacC = \text{Dispersion} + MacD^2 \)

in which

\text{Dispersion} = \frac{\sum_{t>0} (t - MacD)^2 CF_t e^{-\delta t}}{\sum_{t>0} CF_t e^{-\delta t}} = \frac{\sum_{t>0} (t - MacD)^2 CF_t e^{-\delta t}}{P}

\textbf{Note.} For a zero coupon bond the dispersion is zero therefore the Macaulay convexity equals the square of the Macaulay duration. So, for example, a 3-year zero coupon bond has Macaulay convexity of 9.
**Definition.** If we use the approximate formula for the second derivative

$$f''(x) \approx \frac{f(x + h) + f(x - h) - 2f(x)}{h^2}$$

then we can approximate the convexity by the **Effective Convexity** (EffC).

$$\frac{P''(y_0)}{P(y_0)} \approx \frac{P_+ + P_- - 2P_0}{(\Delta y)^2 P_0}$$

where $P_+$ is the price associated with a larger yield than $y_0$, and $P_-$ is the price associated with a smaller yield than $y_0$.

**Effective Convexity:**

$$EffC = \frac{P_+ + P_- - 2P_0}{(\Delta y)^2 P_0}$$
Theorem. A portfolio duration equals the weighted average of the durations of the individual assets in the portfolio, where the weights are the ratios of the values of the assets by the value of the portfolio.

Proof. Consider a portfolio consisting of \( n \) assets with values \( P_1, \ldots, P_n \). Let \( P \) denote the value of the portfolio. Then

\[
P(y) = P_1(y) + \cdots + P_n(y) \quad \Rightarrow \quad P'(y) = P'_1(y) + \cdots + P'_n(y)
\]

Denote by \( D_1(y), \ldots, D_n(y) \) the modified durations of the assets. Then

\[
\text{negative of Duration} = \frac{P'(y)}{P(y)} = \frac{P'_1(y) + \cdots + P'_n(y)}{P(y)} = \frac{P_1(y) P'_1(y)}{P(y) P_1(y)} + \cdots + \frac{P_n(y) P'_n(y)}{P(y) P_n(y)} = \frac{P_1}{P} (-D_1) + \cdots + \frac{P_n}{P} (-D_n)
\]

By dropping a minus sign:

Modified Duration = \( \frac{P_1}{P} D_1 + \cdots + \frac{P_n}{P} D_n \).

This completes the proof for the modified duration. If the derivative is taken with respect to the continuous rate, the we get a similar identity for the Macaulay duration as well.
**Example (from Exam May 2005).** John purchased three bonds to form a portfolio as follows:

Bond A has semiannual coupons at 4%, a duration of 21.46 years, and was purchased for 980.

Bond B is a 15-year bond with a duration of 12.35 years and was purchased for 1015.

Bond C has a duration of 16.67 years and was purchased for 1000.

Calculate the duration of the portfolio at the time of purchase.

**Solution.**

\[
\text{total price} = 980 + 1015 + 1000 = 2995
\]

\[
\text{duration} = \left(\frac{980}{2995}\right) (21.46) + \left(\frac{1015}{2995}\right) (12.35) + \left(\frac{1000}{2995}\right) (16.67) = 16.77
\]

**Note.** The same argument, but with the second derivative, reveals that:

**Theorem.** A portfolio convexity equals the weighted average of the convexities of the individual assets in the portfolio, where the weights are the ratios of the values of the assets by the value of the portfolio.
**Immunization**
*(section 7.7)*

**Definition.** Payments that a company is required to make are called **liability cash flows**.

**Definition.** The cash flows that investments bring in are called **asset cash flows**.

**Definition.**

\[
\text{Surplus} = \text{present value of assets} \text{ minus} \text{ present value of liabilities}
\]

**Definition. Immunization** is a strategy to make sure that changes in interest rate do not affect the surplus.

**Redington immunization** protects the surplus from small changes in the surplus.

Redington immunization is referred to as immunization.

**Three conditions for (Redington) Immunization.**

**Condition 1.** At yield \(y_0\) we want:

\[
\text{Present value of assets} = \text{Present value of liabilities}
\]

**Condition 2.** At yield \(y_0\) we want

\[
\text{Duration of assets} = \text{Duration of liabilities}
\]

**Condition 3.** At yield \(y_0\) we want

\[
\text{Convexity of assets} > \text{Convexity of liabilities}
\]
**Note.** Let us denote the surplus by $S(y)$:

$$S(y) = PV_A(y) - PV_L(y)$$

Under the condition 2 we have $ModD_A(y_0) = ModD_L(y_0)$. Then these three conditions say:

1. $$S(y_0) = PV_A(y_0) - PV_L(y_0) = 0$$

2. $$S'(y_0) = PV'_A(y_0) - PV'_L(y_0)$$

   $$ = PV_A(y_0)(-ModD_A)(y_0) - PV_L(y_0)(-ModD_L)(y_0)$$

   $$ = 0$$

   The last equality is true because $PV_A(y_0) = PV_L(y_0)$ and $ModD_A(y_0) = ModD_L(y_0)$.

3. $$S''(y_0) = PV''_A(y_0) - PV''_L(y_0) > 0$$

But as we have seen on Calculus (from a test called Second-Order Derivative Test), these conditions are sufficient to have a local minimum at $y_0$ for the function $S(y)$. Therefore, for small changes $\Delta y$ in the yield we have

$$S(y_0 + \Delta y) > \min = S(y_0)$$

Therefore a small change in the yield at that point causes the surplus to be positive which is desirable.

**Exercise 7.18 of the textbook.** An insurance company has committed to make a payment of $100,000 in 5 years. The insurance company can fund this liability only through the purchase of 4-year zero-coupon bonds and 10-year zero coupon bonds. The annual effective
yield for all assets and liabilities is 12%. Determine how much the bank should invest in each bond in order to immunize its position.

**Solution.**

The total amount invested must equal the present value of liability:

\[ PV_A = PV_L = \frac{100,000}{(1.12)^5} = 56742.69 \]

Let \( X \) be the percentage of the amount 56742.69 invested in 4-year bonds and \( 1 - X \) is the percentage invested in 10-year bond.

\[
MacD_A = MacD_L \quad \Rightarrow \quad 4X + (1 - X)10 = 5 \quad \Rightarrow \quad -6X = -5 \quad \Rightarrow \\
X = 0.8833 \quad 1 - X = 0.1667
\]

These percentages have been found by imposing the second condition of immunization. Now we must check whether these percentages satisfy the third conditions.

\[
MacC_L = 5^2 = 25 \\
MacC_A = (0.8833)(4^2) + (0.1667)(10^2) = 30
\]

\[
\Rightarrow \quad MacC_A > MacC_L \quad \checkmark
\]

So, these percentages provide (Redington) immunization. Now we calculate the amounts to be invested in each bond:

**amount invested in 4-year bonds** = \((0.8333)(56742.69) = 47285.58\)

**amount invested in 4-year bonds** = \((0.1667)(56742.69) = 9457.12\)
Full Immunization
(section 7.8)

The Redington immunization protects the surplus for small changes of the yield. The full immunization protects the surplus against large changes in the yield.

Under a special circumstances full immunization occurs. In fact, assume that there is a single liability payable at time $T$, and there are two asset cash flows at times $T_1$ and $T_2$, where

$$0 \leq T_1 < T < T_2$$

Three conditions for Full Immunization.

Condition 1. At yield $y_0$ we want:

$$\text{Present value of assets} = \text{Present value of liabilities}$$

Condition 2. At yield $y_0$ we want

$$\text{Duration of assets} = \text{Duration of liabilities}$$

Condition 3. The asset cash flows occur before and after the liability cash flow, i.e.

$$0 \leq T_1 < T < T_2$$

Note. Full immunization implies Redington immunization.

Example (from the textbook). An insurance company has an obligation to pay $1,000,000 at the end of 10 years. It has a zero coupon bond that matures for $413,947.55 in 5 years, and it has a zero coupon bond that matures for $864,580.82 in 20 years. The current annual effective yield is 10%.
(i) Is the company’s position fully immunized.

(ii) Does the company’s position satisfy the conditions for Redington’s immunization?

(iii) What is the new level of surplus if the interest rate falls to 0%?

(iv) What is the new level of surplus if the interest rate rises to 80%?

**Solution to part (i).**

\[
P_{VA} = \frac{1000,000}{(1.10)^{10}} = 385543.29
\]

\[
P_{VL} = 413947.55 \cdot (1.10)^{5} + 846580.82 \cdot (1.10)^{20} = 385543.29
\]

Therefore:

\[
P_{VA} = P_{VL}.
\]

The first condition is met.

Since there is only one cash flow for liability and that occurs at time 10, we have

\[
MacD_L = 10
\]

(use the formula of duration to see this).

**The Macaulay duration of assets:**

\[
MacD_A = \frac{(413947.55)(1.10)^{-5}(5) + (846580.82)(1.10)^{-20}(20)}{PV_A}
\]

\[
= \frac{(413947.55)(1.10)^{-5}(5) + (846580.82)(1.10)^{-20}(20)}{(413947.55)(1.10)^{-5} + (846580.82)(1.10)^{-20}} = 10
\]

Therefore:

\[
MacD_A = MacD_L.
\]

The second condition is met.

Finally, the relationship between the asset cash flows and liability cash flows is:
So the third condition is also met, and we have full immunization.

**Solution to part (ii).**

Since full immunization implies Redington immunization, the conditions for Redington's immunization are also satisfied.

**Solution to part (iii).**

If the interest rate falls to 0% then the new level of surplus will be:

\[
\text{Surplus} = PV_A - PV_L = \left\{ \frac{413947.55}{(1.00)^5} + \frac{846580.82}{(1.00)^{20}} \right\} - \left\{ \frac{1000000}{(1.00)^{10}} \right\}
\]

\[= 287,528.37 \text{ note that, this is positive}\]

**Solution to part (iv).**

If the interest rate increases to 80% then the new level of surplus will be:

\[
\text{Surplus} = PV_A - PV_L = \left\{ \frac{413947.55}{(1.80)^5} + \frac{846580.82}{(1.80)^{20}} \right\} - \left\{ \frac{1000000}{(1.80)^{10}} \right\}
\]

\[= 19,113.02 \text{ note that, this is positive}\]

**Note.** We saw above how to fully immunize a single liability. If there are multiple liabilities, then we must fully immunize each liability by allocating two assets for each.
Example (from the textbook). A company has the following projected liability cash flows:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability cash flow</td>
<td>179</td>
<td>679</td>
<td>144</td>
<td>3144</td>
<td>824</td>
</tr>
</tbody>
</table>

There are three assets available for investment:

- 2-year bond with annual coupons of 7%
- 4-year bond with annual coupons of 4%
- 5-year bond with annual coupons of 3%

Each bond has a par value of $100. The annual effective yield on all three bonds is 5%.

The company has decided to pursue a dedication strategy. Determine the amount of each bond to be purchased, and calculate the cost of establishing the asset portfolio.

Solution.

Step 1. We first offset the liability at end of year 5. The 5-year bond provides the cash flow $100 + 3 = 103$ at year 5. Now we determine the number of such bonds to meet the liability at that year:

The number of 5-year bonds to purchase is \( \frac{824}{103} = 8.0 \)

Now we subtract the cash flow generated by the 5-year bond:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability cash flow</td>
<td>179</td>
<td>679</td>
<td>144</td>
<td>3144</td>
<td>824</td>
</tr>
<tr>
<td>Cash flow from eight 5-year bonds</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>824</td>
</tr>
<tr>
<td>Net liability remaining</td>
<td>155</td>
<td>655</td>
<td>120</td>
<td>3120</td>
<td>0</td>
</tr>
</tbody>
</table>
**Step 2.** The 4-year bond generates $100 + 4 = 104$ at the end of year 4. The number of 4-year bonds required to cover the liability of 3120 at that time is:

The number of 4-year bonds to purchase is \( \frac{3120}{104} = 30 \)

Now we subtract the cash flow generated by the 4-year bonds:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability cash flow</td>
<td>179</td>
<td>679</td>
<td>144</td>
<td>3144</td>
<td>824</td>
</tr>
<tr>
<td>Cash flow from eight 5-year bonds</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>824</td>
</tr>
<tr>
<td>Net liability remaining</td>
<td>155</td>
<td>655</td>
<td>120</td>
<td>3120</td>
<td>0</td>
</tr>
<tr>
<td>Cash flow from thirty 4-year bonds</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>3120</td>
<td>0</td>
</tr>
<tr>
<td>Net liability remaining</td>
<td>35</td>
<td>535</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Step 3.** The 2-year bond generates $100 + 7 = 107$ at the end of year 2. The number of 2-year bonds required to cover the liability of 3120 at that time is:

The number of 2-year bonds to purchase is \( \frac{535}{107} = 5 \)

Now we subtract the cash flow generated by the 2-year bonds:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability cash flow</td>
<td>179</td>
<td>679</td>
<td>144</td>
<td>3144</td>
<td>824</td>
</tr>
<tr>
<td>Cash flow from eight 5-year bonds</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>824</td>
</tr>
<tr>
<td>Net liability remaining</td>
<td>155</td>
<td>655</td>
<td>120</td>
<td>3120</td>
<td>0</td>
</tr>
<tr>
<td>Cash flow from thirty 4-year bonds</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>3120</td>
<td>0</td>
</tr>
<tr>
<td>Net liability remaining</td>
<td>35</td>
<td>535</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Cash flow from five 2-year bonds</td>
<td>35</td>
<td>535</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Net liability remaining</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Conclusion. The purchase of eight 5-year bonds, thirty 4-year bonds, and five 3-year bonds results in a cash-matched portfolio.

Step 4. Now we determine the cost of asset portfolio:

Price of 2-year bond = \(7a_{\overline{2}|5\%} + \frac{100}{(1.05)^2}\) = 103.7188

Price of 4-year bond = \(4a_{\overline{4}|5\%} + \frac{100}{(1.05)^4}\) = 96.4540

Price of 5-year bond = \(3a_{\overline{5}|5\%} + \frac{100}{(1.05)^5}\) = 91.3410

Cost of establishing portfolio = \((5)(103.7188) + (30)(96.4540) + (8)(91.3410)\) = 4142.94

Example (from Exam May 2005). An insurance company accepts an obligation to pay 10,000 at the end of each year for 2 years. The insurance company purchases a combination of the following two bonds at a total cost of \(X\) in order to exactly match its obligation:
1-year 4% annual coupon bond with a yield rate of 5%
2-year 6% annual coupon bond with a yield rate of 5%

Calculate \(X\)

Solution.

Step 1. We first offset the liability at end of year 2. The 2-year bond provides the cash flow \(100 + 6 = 106\) at year 2. Now we determine the number of such bonds to meet the liability at that year:

The number of 2-year bonds to purchase is \(\frac{10,000}{106} = 94.34\)

Now we subtract the cash flow generated by the 2-year bonds:
Step 2. The 1-year bond generates $100 + 4 = 104$ at the end of year 1. The number of 1-year bonds required to cover the liability of $3120$ at that time is:

The number of 1-year bonds to purchase is \( \frac{9433.99}{104} = 90.71 \)

Now we subtract the cash flow generated by the 1-year bonds:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability cash flow</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>Cash flow from 94.34 2-year bonds</td>
<td>566.04</td>
<td>10,000</td>
</tr>
<tr>
<td>Net liability remaining</td>
<td>9433.96</td>
<td>0</td>
</tr>
<tr>
<td>Cash flow from 90.71 1-year bonds</td>
<td>9433.99</td>
<td>0</td>
</tr>
<tr>
<td>Net liability remaining</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Step 3. Now we determine the cost of asset portfolio:

Price of a 1-year bond = \( \frac{104}{1.05} = 99.047 \)

Price of a 2-year bond = \( 6a_{\overline{2}\text{3%}} + \frac{100}{(1.05)^2} = 6 \left( \frac{1-(1.05)^{-2}}{0.05} \right) + \frac{100}{(1.05)^2} = 101.8594 \)

Cost of establishing portfolio = \( (90.71)(99.047) + (94.34)(101.8594) = 18593.9691 \approx 18594 \)

Question. In the previous example, what is the annual effective yield rate for this investment to exactly match the liabilities?

Solution. The present value of two payments of $10,000$ (which occur at the end of the first
two years) must equal 18594.

\[ 18594 = 10000v + 10000v^2 \quad \Rightarrow \quad 10000v^2 + 10000v - 18594 = 0 \]

\[ v = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10000 + \sqrt{(10000)(10000) - (4)(10000)(18594)}}{2} = 0.9524 \]

\[ i = \frac{1}{v} - 1 = 0.05 = 5\% \]