Convexity
(section 7.5)

The concept of duration was studied when we used a linear approximation for $P(y+\Delta y)$. To achieve a more accurate approximation one needs to employ higher order derivatives in the Taylor expansion. If we stop at the second order derivative, we will have the approximation:

$$P(y_0+\Delta y) \approx P(y_0) + P'(y_0)(\Delta y) + \frac{1}{2}P''(y_0)(\Delta y)^2$$

Then:

$$\frac{P(y_0+\Delta y)-P(y_0)}{P(y_0)} \approx (\Delta y)\frac{P'(y_0)}{P(y_0)} + \frac{(\Delta y)^2}{2}\frac{P''(y_0)}{P(y_0)}$$

$$\%\,\Delta P \approx (\Delta y)\frac{P'(y_0)}{P(y_0)} + \frac{(\Delta y)^2}{2}\frac{P''(y_0)}{P(y_0)}$$

Definition. The term $\frac{P''(y_0)}{P(y_0)}$ is called the **convexity** at y_0.

So:

$$\%\,\Delta P \approx -(\Delta y)(\text{Duration}) + \frac{(\Delta y)^2}{2}\text{Convexity}$$

Note. The following formula can be used to calculate convexity :

$$v^n = (1+y)^{-n} \qquad \Rightarrow \qquad \frac{d(v^n)}{dy} = -n(1+y)^{-n-1} = -n\,v^{n+1}$$

$$\boxed{\frac{d(v^n)}{dy} = -n\,v^{n+1}}$$

Note. For bonds with fixed cash flows, convexity is positive for all y_0's.

Example (from the study manual). A \$1000 3-year par value bond yields an effective annual interest rate of 6%. Coupons are paid an annual basis at a rate of 5% per year.

Determine the convexity of the bond.

Solution.

$P = 50\, a_{\overline{3}|6\%} + 1000\, v^3 = 973.27$

$P(y) = 50(v + v^2) + 1050\, v^3$

$P' = 50(-v^2 - 2v^3) + 1050\,(-3v^4)$

$P'' = 50(2v^3 + 6v^4) + 1050\,(12v^5) = \text{at } 6\% = 9,737.04$

$$\text{Convexity} = \frac{P''}{P} = \frac{9,737.04}{973.27} = 10.00$$

Note. Macaulay convexity (MacC) , which is $\frac{\frac{d^2P}{d\delta^2}}{P}$, has a simpler formula and is more widely used. In fact:

$$P = \sum_{t>0} CF_t\, e^{-\delta t} \quad \Rightarrow \quad \frac{dP}{d\delta} = \sum_{t>0} (-t) CF_t\, e^{-\delta t} \quad \Rightarrow \quad \frac{d^2P}{d\delta^2} = \sum_{t>0}^{T} t^2 CF_t\, e^{-\delta t}$$

Dividing it by P we get the Macaulay Duration:

Macaulay Convexity:

If a bond with fixed cash flows has a continuously compounded yield of δ , then its Macaulay Convexity is:

$$MacC = \frac{\sum_{t>0} t^2 CF_t\, e^{-\delta t}}{P}$$

Definition. **Dispersion** is defined as the following weighted average:

$$\text{Dispersion} = \frac{\sum_{t>0} (t - MacD)^2 CF_t \, e^{-\delta t}}{\sum_{t>0} CF_t \, e^{-\delta t}}$$

Note.

$$\begin{aligned}
\text{Dispersion} &= \frac{\sum_{t>0} \left\{ t^2 - 2t \, MacD + MacD^2 \right\} CF_t \, e^{-\delta t}}{P} \\
&= \frac{\sum_{t>0} t^2 \, CF_t \, e^{-\delta t} - 2MacD \sum_{t>0} t \, CF_t \, e^{-\delta t} + MacD^2 \sum_{t>0} CF_t \, e^{-\delta t}}{P} \\
&= MacC - 2MacD^2 + MacD^2 \\
&= MacC - MacD^2
\end{aligned}$$

Dispersion:

If a bond with fixed cash flows has a continuously compounded yield of δ , then its Macaulay Convexity can be written as:

$MacC = \text{Dispersion} + MacD^2$

in which

$$\text{Dispersion} = \frac{\sum_{t>0} (t - MacD)^2 CF_t \, e^{-\delta t}}{\sum_{t>0} CF_t \, e^{-\delta t}} = \frac{\sum_{t>0} (t - MacD)^2 CF_t \, e^{-\delta t}}{P}$$

Note. For a zero coupon bond the dispersion is zero therefore the Macaulay convexity equals the square of the Macaulay duration. So, for example, a 3-year zero coupon bond has Macaulay convexity of 9.

Definition. If we use the approximate formula for the second derivative

$$f''(x) \approx \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

then we can approximate the convexity by the **Effective Convexity** (EffC).

$$\frac{P''(y_0)}{P(y_0)} \approx \frac{P_+ + P_- - 2P_0}{(\Delta y)^2 P_0}$$

where P_+ is the price associated with a larger yield than y_0 , and P_- is the price associated with a smaller yield than y_0.

Effective Convexity:

$$EffC = \frac{P_+ + P_- - 2P_0}{(\Delta y)^2 P_0}$$

Portfolio Duration and convexity
(from section 7.6)

Theorem. A portfolio duration equals the weighted average of the durations of the individual assets in the portfolio, where the weights are the ratios of the values of the assets by the value of the portfolio.

Proof. Consider a portfolio consisting of n asses with values P_1 , ... , P_n. Let P denote the value of the portfolio. Then

$$P(y) = P_1(y) + \cdots + P_n(y) \qquad \Rightarrow \qquad P'(y) = P_1'(y) + \cdots + P_n'(y)$$

Denote by $D_1(y)$, ... , $D_n(y)$ the modified durations of the assets. Then

$$\begin{aligned} \text{negative of Duration} &= \frac{P'(y)}{P(y)} \\ &= \frac{P_1'(y) + \cdots + P_n'(y)}{P(y)} \\ &= \frac{P_1(y)}{P(y)}\frac{P_1'(y)}{P_1(y)} + \cdots + \frac{P_n(y)}{P(y)}\frac{P_n'(y)}{P_n(y)} \\ &= \frac{P_1}{P}\,(-D_1) + \cdots + \frac{P_n}{P}\,(-D_n) \end{aligned}$$

By dropping a minus sign:

$$\text{Modified Duration} = \frac{P_1}{P}\,D_1 + \cdots + \frac{P_n}{P}\,D_n.$$

This completes the proof for the modified duration. If the derivative is taken with respect to the continuous rate, the we get a similar identity for the Macaulay duration as well.

Example (from Exam May 2005). John purchased three bonds to form a portfolio as follows:

Bond A has semiannual coupons at 4% , a duration of 21.46 years, and was purchased for 980.

Bond B is a 15-year bond with a duration of 12.35 years and was purchased for 1015.

Bond C has a duration of 16.67 years and was purchased for 1000.

Calculate the duration of the portfolio at the time of purchase.

Solution.

total price = 980+1015+1000=2995

duration= $\left(\frac{980}{2995}\right)(21.46) + \left(\frac{1015}{2995}\right)(12.35) + \left(\frac{1000}{2995}\right)(16.67) = 16.77$

Note . The same argument, but with the second derivative, reveals that:

Theorem. A portfolio convexity equals the weighted average of the convexities of the individual assets in the portfolio, where the weights are the ratios of the values of the assets by the value of the portfolio.

Immunization
(section 7.7)

Definition. Payments that a company is required to make are called **liability cash flows**.

Definition. The cash flows that investments bring in are called **asset cash flows**.

Definition.

$$\textbf{Surplus} = \text{present value of assets } \textbf{minus} \text{ present value of liabilities}$$

Definition. **Immunization** is a strategy to make sure that changes in interest rate do not affect the surplus.

Redington immunization protects the surplus from small changes in the surplus.

Redington immunization is referred to as immunization.

Three conditions for (Redington) Immunization.

Condition 1. At yield y_0 we want:

$$\text{Present value of assets} = \text{Present value of liabilities}$$

Condition 2. At yield y_0 we want

$$\text{Duration of assets} = \text{Duration of liabilities}$$

Condition 3. At yield y_0 we want

$$\text{Convexity of assets} > \text{Convexity of liabilities}$$

Note. Let us denote the surplus by $S(y)$:

$$S(y) = PV_A(y) - PV_L(y)$$

Under the condition 2 we have $ModD_A(y_0) = ModD_L(y_0)$. Then these three conditions say:

1.
$$S(y_0) = PV_A(y_0) - PV_L(y_0) = 0$$

2.
$$\begin{aligned} S'(y_0) &= PV_A'(y_0) - PV_L'(y_0) \\ &= PV_A(y_0)(-ModD_A)(y_0) - PV_L(y_0)(-ModD_L)(y_0) \\ &= 0 \end{aligned}$$

thae last equality is true because $PV_A(y_0) = PV_L(y_0)$ and $ModD_A(y_0) = ModD_L(y_0)$.

3.
$$S''(y_0) = PV_A''(y_0) - PV_L''(y_0) > 0$$

But as we have seen on Calculus (from a test called Second-Order Derivative Test), these conditions are sufficient to have a local minimum at y_0 for the function $S(y)$. Therefore, for small changes Δy in the yield we have

$$S(y_0 + \Delta y) > \min = S(y_0)$$

Therefore a small change in the yield at that point causes the surplus to be positive which is desirable.

Exercise 7.18 of the textbook. An insurance company has committed to make a payment of $100,000 in 5 years. The insurance company can fund this liability only through the purchase of 4-year zero-coupon bonds and 10-year zero coupon bonds. The annual effective yield for all assets and liabilities is 12%. Determine how much the bank should invest in each bond in order to immunize its position.

Solution.

The total amount invested must equal the present value of liability:

$$PV_A = PV_L = \frac{100,000}{(1.12)^5} = 56742.69$$

Let X be the percentage of the amount 56742.69 invested in 4-year bonds and $1 - X$ is the percentage invested in 10-year bond.

$$MacD_A = MacD_L \quad \Rightarrow \quad 4X + (1 - X)10 = 5 \quad \Rightarrow \quad -6X = -5 \quad \Rightarrow$$

$$X = 0.8833 \quad 1 - X = 0.1667$$

These percentages have been found by imposing the second condition of immunization. Now we must check whether these percentages satisfy the third conditions?.

$$\left.\begin{array}{c} MacC_L = 5^2 = 25 \\ MacC_A = (0.8833)(4^2) + (0.1667)(10^2) = 30 \end{array}\right\} \quad \Rightarrow \quad MacC_A > MacC_L \qquad \checkmark$$

So, these percentages provide (Redington) immunization. Now we calculate the amounts to be invested in each bond:

amount invested in 4-year bonds $= (0.8333)(56742.69) = 47285.58$

amount invested in 4-year bonds $= (0.1667)(56742.69) = 9457.12$

Full Immunization
(section 7.8)

The Redington immunization protects the surplus for small changes of the yield. The **full immunization** protects the surplus against large changes in the yield.

Under a special circumstances full immunization occurs. In fact, assume that there is a single liability payable at time T , and there are two asset cash flows at times T_1 and T_2 , where

$$0 \le T_1 < T < T_2$$

Three conditions for Full Immunization.

Condition 1. At yield y_0 we want:

$$\text{Present value of assets} = \text{Present value of liabilities}$$

Condition 2. At yield y_0 we want

$$\text{Duration of assets} = \text{Duration of liabilities}$$

Condition 3. The asset cash flows occur before and after the liability cash flow , i.e.

$$0 \le T_1 < T < T_2$$

Note. Full immunization implies Redington immunization.

Example (from the textbook). An insurance company has an obligation to pay \$1000,000 at the end of 10 years. It has a zero coupon bond that matures for \$413,947.55 in 5 years, and it has a zero coupon bond that matures for \$864,580.82 in 20 years. The current annual effective yield is 10%.

(i) Is the company's position fully immunized.

(ii) Does the company's position satisfy the conditions for Redington's immunization?.

(iii) What is the new level of surplus if the interest rate falls to 0%?

(iv) What is the new level of surplus if the interest rate rises to 80% ?

Solution to part (i).

$$PV_L = \frac{1000,000}{(1.10)^{10}} = 385543.29$$

$$PV_A = \frac{413947.55}{(1.10)^5} + \frac{846580.82}{(1.10)^20} = 385543.29$$

Therefore:

$PV_A = PV_L$. The first condition is met.

Since there is only one cash flow for liability and that occurs at time 10, we have $MacD_L = 10$ (use the formula of duration to see this).

The Macaulay duration of assets:

$$MacD_A = \frac{(413947.55)(1.10)^{-5}(\mathbf{5}) + (846580.82)(1.10)^{-20}(\mathbf{20})}{PV_A}$$

$$= \frac{(413947.55)(1.10)^{-5}(\mathbf{5}) + (846580.82)(1.10)^{-20}(\mathbf{20})}{(413947.55)(1.10)^{-5} + (846580.82)(1.10)^{-20}} = 10$$

Therefore:

$MacD_A = MacD_L$. The second condition is met.

Finally, the relationship between the asset cash flows and liability cash flows is:

$$\mathbf{5} < \mathbf{10} < \mathbf{20}$$

So the third condition is also met , and we have full immunization.

Solution to part (ii).

Since full immunization implies Redington immunization, the conditions for Redington's immunization are also satisfied.

Solution to part (iii).

If the interest rate falls to 0% then the new level of surplus will be:

$$\text{Surplus} = PV_A - PV_L = \left\{\frac{413947.55}{(1.00)^5} + \frac{846580.82}{(1.00)^{20}}\right\} - \left\{\frac{1000,000}{(1.00)^{10}}\right\}$$

$= \$287,528.37$ note that, this is positive

Solution to part (iv).

If the interest rate increases to 80% then the new level of surplus will be:

$$\text{Surplus} = PV_A - PV_L = \left\{\frac{413947.55}{(1.80)^5} + \frac{846580.82}{(1.80)^20}\right\} - \left\{\frac{1000,000}{(1.80)^{10}}\right\}$$

$= \$19,113.02$ note that, this is positive

Note. We saw above how to fully immunize a single liability. If there are multiple liabilities, then we must fully immunize each liability by allocating two assets for each.

Dedication (portfolio matching)
(section 7.9)

Example (from the textbook). A company has the following projected liability cash flows:

Year	1	2	3	4	5
Liability cash flow	179	679	144	3144	824

There are three assets available for investment:

- 2-year bond with annual coupons of 7%
- 4-year bond with annual coupons of 4%
- 5-year bond with annual coupons of 3%

Each bond has a par value of \$100. The annual effective yield on all three bonds is 5%. The company has decided to pursue a dedication strategy. Determine the amount of each bond to be purchased, and calculate the cost of establishing the asset portfolio.

Solution.

Step 1. We first offset the liability at end of year 5. The 5-year bond provides the cash flow $100 + 3 = 103$ at year 5. Now we determine the number of such bonds to meet the liability at that year:

$$\text{The number of 5-year bonds to purchase is } \tfrac{824}{103} = 8.0$$

Now we subtract the cash flow generated by the 5-year bond:

Year	1	2	3	4	5
Liability cash flow	179	679	144	3144	824
Cash flow from eight 5-year bonds	24	24	24	24	824
Net liability remaining	155	655	120	3120	0

Step 2. The 4-year bond generates $100 + 4 = 104$ at the end of year 4. The number of 4-year bonds required to cover the liability of 3120 at that time is:

$$\text{The number of 4-year bonds to purchase is } \frac{3120}{104} = 30$$

Now we subtract the cash flow generated by the 4-year bonds:

Year	1	2	3	4	5
Liability cash flow	179	679	144	3144	824
Cash flow from eight 5-year bonds	24	24	24	24	824
Net liability remaining	155	655	120	3120	0
Cash flow from thirty 4-year bonds	120	120	120	3120	0
Net liability remaining	35	535	0	0	0

Step 3. The 2-year bond generates $100 + 7 = 107$ at the end of year 2. The number of 2-year bonds required to cover the liability of 3120 at that time is:

$$\text{The number of 2-year bonds to purchase is } \frac{535}{107} = 5$$

Now we subtract the cash flow generated by the 2-year bonds:

Year	1	2	3	4	5
Liability cash flow	179	679	144	3144	824
Cash flow from eight 5-year bonds	24	24	24	24	824
Net liability remaining	155	655	120	3120	0
Cash flow from thirty 4-year bonds	120	120	120	3120	0
Net liability remaining	35	535	0	0	0
Cash flow from five 2-year bonds	35	535	0	0	0
Net liability remaining	0	0	0	0	0

Conclusion. The purchase of eight 5-year bonds, thirty 4-year bonds, and five 3-year bonds results in a cash-matched portfolio.

Step 4. Now we determine the cost of asset portfolio:

Price of 2-year bond $= 7a_{\overline{2}|5\%} + \frac{100}{(1.05)^2} = 103.7188$

Price of 4-year bond $= 4a_{\overline{4}|5\%} + \frac{100}{(1.05)^4} = 96.4540$

Price of 5-year bond $= 3a_{\overline{5}|5\%} + \frac{100}{(1.05)^5} = 91.3410$

Cost of establishing portfolio $= (5)(103.7188) + (30)(96.4540) + (8)(91.3410) = 4142.94$

Example (from Exam May 2005). An insurance company accepts an obligation to pay 10,000 at the end of each year for 2 years. The insurance company purchases a combination of the following two bonds at a total cost of X in order to exactly match its obligation:
1-year 4% annual coupon bond with a yield rate of 5%
2-year 6% annual coupon bond with a yield rate of 5%

Calculate X

Solution.

Step 1. We first offset the liability at end of year 2. The 2-year bond provides the cash flow $100 + 6 = 106$ at year 2. Now we determine the number of such bonds to meet the liability at that year:

$$\text{The number of 2-year bonds to purchase is } \frac{10,000}{106} = 94.34$$

Now we subtract the cash flow generated by the 2-year bonds:

Year	1	2
Liability cash flow	1,0000	1,0000
Cash flow from 94.34 2-year bonds	566.04	10,000
Net liability remaining	9433.96	0

Step 2. The 1-year bond generates $100 + 4 = 104$ at the end of year 1. The number of 1-year bonds required to cover the liability of 3120 at that time is:

$$\text{The number of 1-year bonds to purchase is } \frac{9433.99}{104} = 90.71$$

Now we subtract the cash flow generated by the 1-year bonds:

Year	1	2
Liability cash flow	1000	1000
Cash flow from 94.34 2-year bonds	566.04	10,000
Net liability remaining	9433.96	0
Cash flow from 90.71 1-year bonds	9433.99	0
Net liability remaining	0	0

Step 3. Now we determine the cost of asset portfolio:

Price of a 1-year bond $= \frac{104}{1.05} = 99.047$

Price of a 2-year bond $= 6a_{\overline{2}|5\%} + \frac{100}{(1.05)^2} = 6\left(\frac{1-v^2}{i}\right) = 6\left(\frac{1-(1.05)^{-2}}{0.05}\right) + \frac{100}{(1.05)^2} = 101.8594$

Cost of establishing portfolio $= (90.71)(99.047) + (94.34)(101.8594) = 18593.9691 \approx 18594$

Question . In the previous example, what is the annual effective yield rate for this investment to exactly match the liabilities?.

Solution . The present value of two payments of 10,000 (which occur at the end of the first two years) must equal 18594.

$$18594 = 10000v + 10000v^2 \qquad \Rightarrow \qquad 10000v^2 + 10000v - 18594 = 0$$

$$v = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10000 + \sqrt{(10000)(10000) - (4)(10000)(18594)}}{2} = 0.9524$$

$$i = \frac{1}{v} - 1 = 0.05 = 5\%$$