Bühlmann Model

In this model we have an independent identically distributed process $\{X_1, \ldots, X_N, X_{N+1}, \ldots.\}$ with common mean and variance:

$$\text{Hypothetical Mean}: \quad \mu(\theta) = E(X_1|\theta) = E(X_2|\theta) = \cdots$$

$$\text{Process Variance}: \quad \sigma^2(\theta) = \mathrm{Var}(X_1|\theta) = \mathrm{Var}(X_2|\theta) = \cdots$$

The portion $\{X_1, \ldots, X_N\}$ is used to forecast the future outcomes $\{X_{N+1}, X_{n+1}, \ldots.\}$. Now we define the following quantities:

(1) Population mean: $\mu = \mathrm{E}[\mu(\theta)] = \mathrm{E}[\mathrm{E}[X_t|\theta]]$

(2) Expected Value of Process Variance: $\mathrm{EPV} = \mathrm{E}[\sigma^2(\theta)] = \mathrm{E}\Big[\mathrm{Var}[X_t|\theta]\Big]$

(3) Variance of Hypothetical Means: $\mathrm{VHM} = \mathrm{Var}[\mu(\theta)] = \mathrm{E}\Big[(\mu(\theta) - \mu)^2\Big]$

If no prior information is available, then the population mean is used as an estimate for the expected values of the X_t's.

Example (from the Dean's note). The number of claims X_t during the t-th period for a risk has a Poisson distribution with parameter θ:

$$P[X_t = x] = \frac{\theta^x e^\theta}{x!}$$

The risk was selected at random from a population for which θ is uniformly distributed over the interval $[0,1]$. It will be assumed that θ is constant through time for each risk.

(1) Hypothetical mean for risk with parameter θ is $\mu(\theta) = \mathrm{E}[X_t|\theta] = \theta$ because the mean of the Poisson random variable is the parameter θ.

(2) Process variance for risk with parameter θ is

$$\sigma^2(\theta) = \mathrm{Var}[X_t|\theta] = \theta$$

because the variance equals the parameter θ for the Poisson.

(3) Variance of the Hypothetical Means (VHM) is

$$\text{Var}\Big(E[X_t|\theta]\Big) = \text{Var}(\theta) = E(\theta^2) - E(\theta)^2 = \int_0^1 \theta^2 d\theta - \left(\int_0^1 \theta d\theta\right)^2 = \frac{1}{12}$$

(4) Expected Value of the Process Variance (EPV) is

$$E\Big[\text{Var}(X_\theta|\theta)\Big] = E[\theta] = \int_0^1 \theta d\theta = \frac{1}{2}$$

(5) Unconditional Variance (or total variance) is

$$Var[X_\theta] = VHM + EPV = \frac{1}{12} + \frac{1}{2} = \frac{7}{12}$$

Derivation of the Credibility factor in Bühlmann Model

By setting $\bar{X} = \frac{X_1 + \cdots + X_N}{N} = \frac{1}{N}\sum_{i=1}^{N} X_i$ we have

$$E(\bar{X}|\theta) = E\left[\frac{1}{N}\sum_{i=1}^{N} X_i|\theta\right] = \frac{1}{N}\sum_{i=1}^{N} E(X_i|\theta) = \frac{1}{N}\sum_{i=1}^{N}\mu(\theta) = \mu(\theta)$$

So , in other words , $\bar{X}$ is an unbiased estimator for $\mu(\theta)$. Now we seek a and b so as to minimize the expected value:

$$\min\ E\left[a + b\bar{X} - \mu(\theta)\right]^2$$

where the expectation is taking with respect to the joint distribution of $(X_1, \ldots, X_N, \theta)$.

For simplicity, set

$$Y = \bar{X} - \mu(\theta)$$

Then $\bar{X} = Y + \mu(\theta)$ and of course we have

$$E(Y|\theta) = E(\bar{X}|\theta) - E(\mu(\theta)|\theta) = E(\bar{X}|\theta) - \mu(\theta) = 0$$

Now note that

$$\begin{aligned}
\left[a + b\bar{X} - \mu(\theta)\right]^2 &= \left[a + bY + (b-1)\mu(\theta)\right]^2 \\
&= (bY + c(\theta))^2 \qquad c(\theta) = a + (b-1)\mu(\theta) \\
&= b^2Y^2 + 2b\,c(\theta)\,Y + c(\theta)^2
\end{aligned}$$

Then

$$E\left[a + b\bar{X} - \mu(\theta)\right]^2 = b^2E(Y^2) + 2bE\left[c(\theta)Y\right] + E\left[c(\theta)^2\right] \qquad (1)$$

But:

$$E\left[c(\theta)Y\right] = E\left[E\left[c(\theta)Y|\theta\right]\right] = E\left[c(\theta)E\left[Y|\theta\right]\right] = E\left[c(\theta)\,\text{zero}\right] = E(\text{zero}) = 0$$

So the equality (1) reduces to

$$E\left[a + b\bar{X} - \mu(\theta)\right]^2 = b^2E(Y^2) + E\left[c(\theta)^2\right] \qquad (2)$$

To minimize this , we must set the partial derivative of it equal to zero:

$$\frac{\partial}{\partial a} = 2E\left[c(\theta)\,\frac{\partial c(\theta)}{\partial a}\right] = 2E\Big(c(\theta)\Big) = 2\Big\{a + (b-1)E[\mu(\theta)]\Big\} = 2\Big\{a + (b-1)\mu\Big\}$$

Then

$$\text{if } \frac{\partial}{\partial a} = 0 \quad \Rightarrow \quad a = (1-b)\mu$$

<u>**Next Step**</u>. Using the equality $\mathrm{a} = (1-\mathrm{b})\mu$ we can now write the right-hand side of equation (2) as

$$\begin{aligned} \mathrm{b}^2\mathrm{E}(\mathrm{Y}^2) + \mathrm{E}\Big[(1-\mathrm{b})^2(\mu(\theta)-\mu)^2\Big] &= \mathrm{b}^2\mathrm{E}(\mathrm{Y}^2) + (1-\mathrm{b})^2\mathrm{E}\Big[(\mu(\theta)-\mu)^2\Big] \\ &= \mathrm{b}^2\mathrm{E}(\mathrm{Y}^2) + (1-\mathrm{b})^2\mathrm{Var}(\mu(\theta)) \\ &= \mathrm{b}^2\mathrm{E}(\mathrm{Y}^2) + (1-\mathrm{b})^2\mathrm{VHM} \qquad (3) \end{aligned}$$

Further note that

$$\begin{aligned} \mathrm{E}(\mathrm{Y}^2) = \mathrm{E}\Big\{\mathrm{E}[\mathrm{Y}^2|\theta]\Big\} &= \mathrm{E}\Big\{\mathrm{E}\Big[(\bar{\mathrm{X}}-\mu(\theta))^2|\theta\Big]\Big\} \\ &= \mathrm{E}\Big\{\mathrm{Var}\Big[\bar{\mathrm{X}}|\theta\Big]\Big\} = \mathrm{E}\Big\{\tfrac{1}{\mathrm{N}}\mathrm{Var}\Big[\mathrm{X}_1|\theta\Big]\Big\} \\ &= \tfrac{1}{\mathrm{N}}\mathrm{E}\Big\{\mathrm{Var}\Big[\mathrm{X}_1|\theta\Big]\Big\} = \tfrac{1}{\mathrm{N}}\mathrm{EPV} \qquad (4) \end{aligned}$$

Putting this into (3) , the right-hand side of (3) reads:

$$b^2\,\frac{EPV}{N} + (1-b)^2\,VHM$$

Now differentiate this with respect to b and set it equal to zero:

$$\frac{\partial}{\partial b} = 0 \quad \Rightarrow \quad 2b\frac{EPV}{N} - 2(1-b)VHM$$

$$\Rightarrow \quad b = \frac{VHM}{VHM + \frac{EPV}{N}} = \frac{N}{N + \frac{EPV}{VHM}} = \frac{N}{N+K} \qquad \text{where} \qquad K = \frac{EPV}{VHM}$$

This quantity is denoted by Z:

$$Z = \frac{VHM}{VHM + \frac{EPV}{N}}$$

Then

$$a = (1-b)\mu = (1-Z)\mu$$

Then our estimate for $\mu(\theta)$ will be

$$\hat{\mu}(\theta) = a + b\bar{X} = (1-Z)\mu + Z\bar{X}$$

Note. As we saw in the calculations in (4) we have:

$$E\left\{\text{Var}\left[\bar{X}|\theta\right]\right\} = \frac{1}{N}EPV$$

Also:

$$\text{Var}\left\{E\left[\bar{X}|\theta\right]\right\} = \text{Var}\left\{E\left[\frac{1}{N}\sum_{i=1}^{N}X_i|\theta\right]\right\} = \text{Var}\left\{\frac{1}{N}\sum_{i=1}^{N}E\left[X_i|\theta\right]\right\} = \text{Var}\left\{\frac{1}{N}\sum_{i=1}^{N}\mu(\theta)\right\} = \text{Var}(\mu(\theta)) = VHM$$

Now by adding up these expressions , we get:

$$E\left\{\text{Var}\left[\bar{X}|\theta\right]\right\} + \text{Var}\left\{E\left[\bar{X}|\theta\right]\right\} = \frac{EPV}{N} + VHM \quad \Rightarrow \quad \text{Var}(\bar{X}) = \frac{EPV}{N} + VHM$$

$$Z = \frac{VHM}{VHM + \frac{EPV}{N}} = \frac{\text{Var}(\mu(\theta))}{\text{Var}(\bar{X})} = \frac{\text{Variance of the Hypothetical Means}}{\text{Total Variance of the Estimator } \bar{X}}$$

Also note that

$$K = \frac{E(\text{Var}[X|\theta])}{\text{Var}(E[X|\theta])}$$

Example (from the Dean's notes). Two risks have the following severity distributions and that Risk 1 is twice as likely to be observed as Risk 2.

Amount of Claim	Probability of Claim Amount of Risk 1	Probability of Claim Amount of Risk 2
250	0.5	0.7
2500	0.3	0.2
60000	0.2	0.1

A claim of 250 is observed. Determine the Bühlmann credibility estimate of the second claim amount from the same risk.

Solution

Let us denote the claim amount by X.

Step 1. Calculate the variance of the hypothetical means :

$$E[X|\text{Risk 1}] = (0.5)(250) + (0.3)(2500) + (0.2)(60000) = 12875$$

$$E[X|\text{Risk 2}] = (0.7)(250) + (0.2)(2500) + (0.1)(60000) = 6675$$

$$E[X] = (\frac{2}{3})(12875) + (\frac{1}{3})(6675) = 10808.33$$

$$VHM = (\frac{2}{3})(12875 - 10808.33)^2 + (\frac{1}{3})(6675 - 10808.33)^2 = 8542,222.2$$

Step 2. Calculate the expected value of the process variance :

$$\text{Var}[X|\text{Risk 1}] = (0.5)(250 - 12875)^2 + (0.3)(2500 - 12875)^2 + (0.2)(60000 - 12875)^2 = 55,6140,625.0$$

$$\text{Var}[X|\text{Risk 2}] = (0.7)(250 - 6,675)^2 + (0.2)(2500 - 6675)^2 + (0.1)(60000 - 6675)^2 = 316,738,125.0$$

$$EPV = (\frac{2}{3})(556,140,625.0) + (\frac{1}{3})(316,738,125.0) = 476,339,791.7$$

$$K = \frac{EPV}{VHM} = \frac{476,339,791.7}{8,542,222.2} = 55.76$$

$$Z = \frac{N}{N+K} = \frac{1}{1+55.76} = \frac{1}{56.76}$$

$$\text{Bühlmann credibility estimate} = (\frac{1}{56.76})(250) + (\frac{55.76}{56.76})(10,808.33) = 10,622$$

Example $*$. You are given the following:

(i) The number of claims made by an individual insured follows a Poisson distribution.

(ii) The expected number of claims, λ, for insureds in the population has the probability density function

$$f(\lambda) = 4\lambda^{-5} \qquad \text{for} \quad 1 \le \lambda < \infty$$

Determine the value of the Bühlmann k used for estimating the expected number of claims for an individual insured.

Solution. Here X denotes the number of claims.

$$E[X|\lambda] = E(\text{Poisson}(\lambda)) = \lambda$$

$$E(\lambda^2) = 4\int_1^{\infty} \lambda^2 \lambda^{-5}\, d\lambda = 4\int_1^{\infty} \lambda^{-3}\, d\lambda = \frac{4}{-2}\lambda^{-2}\Big]_1^{\infty} = 2$$

$$E(\lambda) = 4\int_1^{\infty} \lambda \lambda^{-5}\, d\lambda = 4\int_1^{\infty} \lambda^{-4}\, d\lambda = \frac{4}{-3}\lambda^{-3}\Big]_1^{\infty} = \frac{4}{3}$$

$$\text{Var}(E[X|\lambda]) = \text{Var}(\lambda) = E(\lambda^2) - E(\lambda)^2 = 2 - \frac{16}{9} = \frac{2}{9}$$

$$\text{Var}[X|\lambda] = \text{Var}(\text{Poisson}(\lambda)) = \lambda$$

$$E(\text{Var}[X|\lambda]) = E(\lambda) = \frac{4}{3}$$

$$K = \frac{E(\text{Var}[X|\lambda])}{\text{Var}(E[X|\lambda])} = \frac{\frac{4}{3}}{\frac{2}{9}} = 6$$